

CHAPTER - 3

TRIGONOMETRIC FUNCTIONS

Exercise 3.1

Question 1:

Find the radian measures corresponding to the following degree measures:

- (i) 25° (ii) $-47^\circ 30'$ (iii) 240° (iv) 520°

Answer 1:

- (i) 25°

We know that $180^\circ = \pi$ radian

$$\therefore 25^\circ = \frac{\pi}{180} \times 25 \text{ radian} = \frac{5\pi}{36} \text{ radian}$$

- (ii) $-47^\circ 30'$

$$-47^\circ 30' = -47\frac{1}{2}$$

$$= -\frac{95}{2} \text{ degree}$$

Since $180^\circ = \pi$ radian

$$\frac{-95}{2} \text{ degree} = \frac{\pi}{180} \times \left(\frac{-95}{2}\right) \text{ radian} = \left(\frac{-19}{36 \times 2}\right) \pi \text{ radian} = \frac{-19}{72} \pi \text{ radian}$$

$$\therefore -47^\circ 30' = \frac{-19}{72} \pi \text{ radian}$$

(iii) 240°

We know that $180^\circ = \pi$ radian

$$\therefore 240^\circ = \frac{\pi}{180} \times 240 \text{ radian} = \frac{4}{3} \pi \text{ radian}$$

(iv) 520°

We know that $180^\circ = \pi$ radian

$$\therefore 520^\circ = \frac{\pi}{180} \times 520 \text{ radian} = \frac{26\pi}{9} \text{ radian}$$

Question 2:

Find the degree measures corresponding to the following radian measures

(Use $\pi = \frac{22}{7}$)

(i) $\frac{11}{16}$

(ii) -4

(iii) $\frac{5\pi}{3}$

(iv) $\frac{7\pi}{6}$

Answer 2:

(i) $\frac{11}{16}$

We know that π radian = 180°

$$\begin{aligned}
\therefore \frac{11}{16} \text{ radian} &= \frac{180}{\pi} \times \frac{11}{16} \text{ deg ree} = \frac{45 \times 11}{\pi \times 4} \text{ deg ree} \\
&= \frac{45 \times 11 \times 7}{22 \times 4} \text{ deg ree} = \frac{315}{8} \text{ deg ree} \\
&= 39 \frac{3}{8} \text{ deg ree} \\
&= 39^\circ + \frac{3 \times 60}{8} \text{ min utes} \quad [1^\circ = 60'] \\
&= 39^\circ + 22' + \frac{1}{2} \text{ min utes} \\
&= 39^\circ 22' 30'' \quad [1' = 60'']
\end{aligned}$$

(ii) - 4

We know that π radian = 180°

$$\begin{aligned}
-4 \text{ radian} &= \frac{180}{\pi} \times (-4) \text{ deg ree} = \frac{180 \times 7(-4)}{22} \text{ deg ree} \\
&= \frac{-2520}{11} \text{ deg ree} = -229 \frac{1}{11} \text{ deg ree} \\
&= -229^\circ + \frac{1 \times 60}{11} \text{ min utes} \quad [1^\circ = 60'] \\
&= -229^\circ + 5' + \frac{5}{11} \text{ min utes} \\
&= -229^\circ 5' 27'' \quad [1' = 60'']
\end{aligned}$$

(iii) $\frac{5\pi}{3}$

We know that π radian = 180°

$$\therefore \frac{5\pi}{3} \text{ radian} = \frac{180}{\pi} \times \frac{5\pi}{3} \text{ deg ree} = 300^\circ$$

(iv) $\frac{7\pi}{6}$

We know that π radian = 180°

$$\therefore \frac{7\pi}{6} \text{ radian} = \frac{180}{\pi} \times \frac{7\pi}{6} = 210^\circ$$

Question 3:

A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

Answer 3:

Number of revolutions made by the wheel in 1 minute = 360

$$\therefore \text{Number of revolutions made by the wheel in 1 second} = \frac{360}{60} = 6$$

In one complete revolution, the wheel turns an angle of 2π radian.

Hence, in 6 complete revolutions, it will turn an angle of $6 \times 2\pi$ radian, i.e., 12π radian

Thus, in one second, the wheel turns an angle of 12π radian.

Question 4:

Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm.

$$\left(\text{Use } \pi = \frac{22}{7} \right)$$

Answer 4:

We know that in a circle of radius r unit, if an arc of length l unit subtends an angle θ radian at the centre, then

$$\theta = \frac{l}{r}$$

Therefore, for $r = 100$ cm, $l = 22$ cm, we have

$$\begin{aligned}\theta &= \frac{22}{100} \text{ radian} = \frac{180}{\pi} \times \frac{22}{100} \text{ deg ree} = \frac{180 \times 7 \times 22}{22 \times 100} \text{ deg ree} \\ &= \frac{126}{10} \text{ deg ree} = 12\frac{3}{5} \text{ deg ree} = 12^\circ 36' \quad [1^\circ = 60']\end{aligned}$$

Thus, the required angle is $12^\circ 36'$.

Question 5:

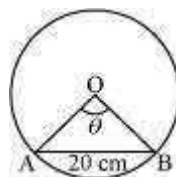
In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

Answer 5:

Diameter of the circle = 40 cm

$$\therefore \text{Radius } (r) \text{ of the circle} = \frac{40}{2} \text{ cm} = 20 \text{ cm}$$

Let AB be a chord (length = 20 cm) of the circle.



In $\triangle OAB$, $OA = OB = \text{Radius of circle} = 20 \text{ cm}$

Also, $AB = 20 \text{ cm}$

Thus, $\triangle OAB$ is an equilateral triangle.

$$\therefore \theta = 60^\circ = \frac{\pi}{3} \text{ radian}$$

We know that in a circle of radius r unit, if an arc of length l unit subtends an angle θ

$$\theta = \frac{l}{r}$$

$$\frac{\pi}{3} = \frac{\widehat{AB}}{20} \Rightarrow \widehat{AB} = \frac{20\pi}{3} \text{ cm}$$

Thus, the length of the minor arc of the chord is $\frac{20\pi}{3} \text{ cm}$

Question 6:

If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

Answer 6:

Let the radii of the two circles be r_1 and r_2 . Let an arc of length l subtend an angle of 60° at the centre of the circle of radius r_1 , while let an arc of length l subtend an angle of 75° at the centre of the circle of radius r_2 .

$$\text{Now, } 60^\circ = \frac{\pi}{3} \text{ radian} \quad \text{and } 75^\circ = \frac{5\pi}{12} \text{ radian}$$

We know that in a circle of radius r unit, if an arc of length l unit subtends an angle θ

$$\theta = \frac{l}{r} \text{ or } l = r\theta$$

$$\therefore l = \frac{r_1\pi}{3} \text{ and } l = \frac{r_2 5\pi}{12}$$

$$\Rightarrow \frac{r_1\pi}{3} = \frac{r_2 5\pi}{12}$$

$$\Rightarrow r_1 = \frac{r_2 5}{4}$$

$$\Rightarrow \frac{r_1}{r_2} = \frac{5}{4}$$

Thus, the ratio of the radii is 5:4.

Question 7:

Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length

(i) 10 cm

(ii) 15 cm

(iii) 21 cm

Answer 7:

We know that in a circle of radius r unit, if an arc of length l unit subtends

an angle θ radian at the centre, then $\theta = \frac{l}{r}$

It is given that $r = 75$ cm

(i) Here, $l = 10$ cm

$$\theta = \frac{10}{75} \text{ radian} = \frac{2}{15} \text{ radian}$$

(ii) Here, $l = 15$ cm

$$\theta = \frac{15}{75} \text{ radian} = \frac{1}{5} \text{ radian}$$

(iii) Here, $l = 21$ cm

$$\theta = \frac{21}{75} \text{ radian} = \frac{7}{25} \text{ radian}$$

Exercise 3.2

Question 1:

Find the values of other five trigonometric functions if $\cos x = -\frac{1}{2}$, x lies in third quadrant.

Answer 1:

$$\cos x = -\frac{1}{2}$$

$$\therefore \sec x = \frac{1}{\cos x} = \frac{1}{\left(-\frac{1}{2}\right)} = -2$$

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \sin^2 x = 1 - \left(-\frac{1}{2}\right)^2$$

$$\Rightarrow \sin^2 x = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \sin x = \pm \frac{\sqrt{3}}{2}$$

Since x lies in the 3rd quadrant, the value of $\sin x$ will be negative.

$$\therefore \sin x = -\frac{\sqrt{3}}{2}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)} = -\frac{2}{\sqrt{3}}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)} = \sqrt{3}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\sqrt{3}}$$

Question 2:

Find the values of other five trigonometric functions if $\sin x = \frac{3}{5}$, x lies in second quadrant.

Answer 2:

$$\sin x = \frac{3}{5}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(\frac{3}{5}\right)} = \frac{5}{3}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \cos^2 x = 1 - \sin^2 x$$

$$\Rightarrow \cos^2 x = 1 - \left(\frac{3}{5}\right)^2$$

$$\Rightarrow \cos^2 x = 1 - \frac{9}{25}$$

$$\Rightarrow \cos^2 x = \frac{16}{25}$$

$$\Rightarrow \cos x = \pm \frac{4}{5}$$

Since x lies in the 2nd quadrant, the value of $\cos x$ will be negative

$$\therefore \cos x = -\frac{4}{5}$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{\left(-\frac{4}{5}\right)} = -\frac{5}{4}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(\frac{3}{5}\right)}{\left(-\frac{4}{5}\right)} = -\frac{3}{4}$$

$$\cot x = \frac{1}{\tan x} = -\frac{4}{3}$$

Question 3:

Find the values of other five trigonometric functions if $\cot x = \frac{3}{4}$, x lies in third quadrant.

Answer 3:

$$\cot x = \frac{3}{4}$$

$$\tan x = \frac{1}{\cot x} = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\Rightarrow 1 + \left(\frac{4}{3}\right)^2 = \sec^2 x$$

$$\Rightarrow 1 + \frac{16}{9} = \sec^2 x$$

$$\Rightarrow \frac{25}{9} = \sec^2 x$$

$$\Rightarrow \sec x = \pm \frac{5}{3}$$

Since x lies in the 3rd quadrant, the value of $\sec x$ will be negative.

$$\therefore \sec x = -\frac{5}{3}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\left(-\frac{5}{3}\right)} = -\frac{3}{5}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\Rightarrow \frac{4}{3} = \frac{\sin x}{\left(-\frac{3}{5}\right)}$$

$$\Rightarrow \sin x = \left(\frac{4}{3}\right) \times \left(-\frac{3}{5}\right) = -\frac{4}{5}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = -\frac{5}{4}$$

Question 4:

Find the values of other five trigonometric functions if $\sec x = \frac{13}{5}$, x lies in fourth quadrant.

Answer 4:

$$\sec x = \frac{13}{5}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\left(\frac{13}{5}\right)} = \frac{5}{13}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \sin^2 x = 1 - \left(\frac{5}{13}\right)^2$$

$$\Rightarrow \sin^2 x = 1 - \frac{25}{169} = \frac{144}{169}$$

$$\Rightarrow \sin x = \pm \frac{12}{13}$$

Since x lies in the 4th quadrant, the value of $\sin x$ will be negative.

$$\therefore \sin x = -\frac{12}{13}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(-\frac{12}{13}\right)} = -\frac{13}{12}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(-\frac{12}{13}\right)}{\left(\frac{5}{13}\right)} = -\frac{12}{5}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\left(-\frac{12}{5}\right)} = -\frac{5}{12}$$

Question 5:

Find the values of other five trigonometric functions if $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Answer 5:

$$\tan x = -\frac{5}{12}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\left(-\frac{5}{12}\right)} = -\frac{12}{5}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\Rightarrow 1 + \left(-\frac{5}{12}\right)^2 = \sec^2 x$$

$$\Rightarrow 1 + \frac{25}{144} = \sec^2 x$$

$$\Rightarrow \frac{169}{144} = \sec^2 x$$

$$\Rightarrow \sec x = \pm \frac{13}{12}$$

Since x lies in the 2nd quadrant, the value of $\sec x$ will be negative.

$$\therefore \sec x = -\frac{13}{12}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\left(-\frac{13}{12}\right)} = -\frac{12}{13}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\Rightarrow -\frac{5}{12} = \frac{\sin x}{\left(-\frac{12}{13}\right)}$$

$$\Rightarrow \sin x = \left(-\frac{5}{12}\right) \times \left(-\frac{12}{13}\right) = \frac{5}{13}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(\frac{5}{13}\right)} = \frac{13}{5}$$

Question 6:

Find the value of the trigonometric function $\sin 765^\circ$

Answer 6:

It is known that the values of $\sin x$ repeat after an interval of 2π or 360° .

$$\therefore \sin 765^\circ = \sin(2 \times 360^\circ + 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

Question 7:

Find the value of the trigonometric function $\operatorname{cosec}(-1410^\circ)$

Answer 7:

It is known that the values of $\operatorname{cosec} x$ repeat after an interval of 2π or 360° .

$$\begin{aligned}\therefore \operatorname{cosec}(-1410^\circ) &= \operatorname{cosec}(-1410^\circ + 4 \times 360^\circ) \\ &= \operatorname{cosec}(-1410^\circ + 1440^\circ) \\ &= \operatorname{cosec}30^\circ = 2\end{aligned}$$

Question 8:

Find the value of the trigonometric function $\tan \frac{19\pi}{3}$

Answer 8:

It is known that the values of $\tan x$ repeat after an interval of π or 180° .

$$\therefore \tan \frac{19\pi}{3} = \tan 6\frac{1}{3}\pi = \tan\left(6\pi + \frac{\pi}{3}\right) = \tan \frac{\pi}{3} = \tan 60^\circ = \sqrt{3}$$

Question 9:

Find the value of the trigonometric function $\sin\left(-\frac{11\pi}{3}\right)$

Answer 9:

It is known that the values of $\sin x$ repeat after an interval of 2π or 360° .

$$\therefore \sin\left(-\frac{11\pi}{3}\right) = \sin\left(-\frac{11\pi}{3} + 2 \times 2\pi\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Question 10:

Find the value of the trigonometric function $\cot\left(-\frac{15\pi}{4}\right)$

Answer 10:

It is known that the values of $\cot x$ repeat after an interval of π or 180° .

$$\therefore \cot\left(-\frac{15\pi}{4}\right) = \cot\left(-\frac{15\pi}{4} + 4\pi\right) = \cot\frac{\pi}{4} = 1$$

Exercise 3.3

Question 1:

$$\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$$

Answer 1:

$$\begin{aligned} \text{L.H.S.} &= \sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} \\ &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - (1)^2 \\ &= \frac{1}{4} + \frac{1}{4} - 1 = -\frac{1}{2} \\ &= \text{R.H.S.} \end{aligned}$$

Question 2:

Prove that $2 \sin^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = \frac{3}{2}$

Answer 2:

$$\begin{aligned} \text{L.H.S.} &= 2 \sin^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} \\ &= 2 \left(\frac{1}{2}\right)^2 + \operatorname{cosec}^2 \left(\pi + \frac{\pi}{6}\right) \left(\frac{1}{2}\right)^2 \\ &= 2 \times \frac{1}{4} + \left(-\operatorname{cosec} \frac{\pi}{6}\right)^2 \left(\frac{1}{4}\right) \\ &= \frac{1}{2} + (-2)^2 \left(\frac{1}{4}\right) \\ &= \frac{1}{2} + \frac{4}{4} = \frac{1}{2} + 1 = \frac{3}{2} \\ &= \text{R.H.S.} \end{aligned}$$

Question 3:

Prove that $\cot^2 \frac{\pi}{6} + \operatorname{cosec} \frac{5\pi}{6} + 3 \tan^2 \frac{\pi}{6} = 6$

Answer 3:

$$\begin{aligned} \text{L.H.S.} &= \cot^2 \frac{\pi}{6} + \operatorname{cosec} \frac{5\pi}{6} + 3 \tan^2 \frac{\pi}{6} \\ &= (\sqrt{3})^2 + \operatorname{cosec} \left(\pi - \frac{\pi}{6} \right) + 3 \left(\frac{1}{\sqrt{3}} \right)^2 \\ &= 3 + \operatorname{cosec} \frac{\pi}{6} + 3 \times \frac{1}{3} \\ &= 3 + 2 + 1 = 6 \\ &= \text{R.H.S.} \end{aligned}$$

Question 4:

Prove that $2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3} = 10$

Answer 4:

$$\text{L.H.S.} = 2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3}$$

$$\begin{aligned}
&= 2 \left\{ \sin \left(\pi - \frac{\pi}{4} \right) \right\}^2 + 2 \left(\frac{1}{\sqrt{2}} \right)^2 + 2(2)^2 \\
&= 2 \left\{ \sin \frac{\pi}{4} \right\}^2 + 2 \times \frac{1}{2} + 8 \\
&= 2 \left(\frac{1}{\sqrt{2}} \right)^2 + 1 + 8 \\
&= 1 + 1 + 8 \\
&= 10 \\
&= \text{R.H.S}
\end{aligned}$$

Question 5:

Find the value of:

(i) $\sin 75^\circ$

(ii) $\tan 15^\circ$

Answer 5:

(i) $\sin 75^\circ = \sin (45^\circ + 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

[$\sin (x + y) = \sin x \cos y + \cos x \sin y$]

$$\begin{aligned}
&= \left(\frac{1}{\sqrt{2}} \right) \left(\frac{\sqrt{3}}{2} \right) + \left(\frac{1}{\sqrt{2}} \right) \left(\frac{1}{2} \right) \\
&= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}$$

(ii) $\tan 15^\circ = \tan (45^\circ - 30^\circ)$

$$\begin{aligned}
&= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} && \left[\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \right] \\
&= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\left(\frac{1}{\sqrt{3}}\right)} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} \\
&= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{3+1-2\sqrt{3}}{(\sqrt{3})^2 - (1)^2} \\
&= \frac{4-2\sqrt{3}}{3-1} = 2 - \sqrt{3}
\end{aligned}$$

Question 6:

Prove that: $\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right) - \sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right) = \sin(x+y)$

Answer 6:

$$\begin{aligned}
&\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right) - \sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right) \\
&= \frac{1}{2}\left[2\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)\right] + \frac{1}{2}\left[-2\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)\right] \\
&= \frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4}-x\right) + \left(\frac{\pi}{4}-y\right)\right\} + \cos\left\{\left(\frac{\pi}{4}-x\right) - \left(\frac{\pi}{4}-y\right)\right\}\right] \\
&\quad + \frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4}-x\right) + \left(\frac{\pi}{4}-y\right)\right\} - \cos\left\{\left(\frac{\pi}{4}-x\right) - \left(\frac{\pi}{4}-y\right)\right\}\right] \\
&\quad \left[\begin{array}{l} \because 2\cos A \cos B = \cos(A+B) + \cos(A-B) \\ -2\sin A \sin B = \cos(A+B) - \cos(A-B) \end{array} \right] \\
&= 2 \times \frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4}-x\right) + \left(\frac{\pi}{4}-y\right)\right\}\right] \\
&= \cos\left[\frac{\pi}{2} - (x+y)\right] \\
&= \sin(x+y) \\
&= \text{R.H.S}
\end{aligned}$$

Question 7:

Prove that:
$$\frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2$$

Answer 7:

It is known that

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \text{ and } \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\text{L.H.S.} = \frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \frac{\left(\frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4} \tan x}\right)}{\left(\frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4} \tan x}\right)} = \frac{\left(\frac{1 + \tan x}{1 - \tan x}\right)}{\left(\frac{1 - \tan x}{1 + \tan x}\right)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2 = \text{R.H.S.}$$

Question 8:

Prove that
$$\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = \cot^2 x$$

Answer 8:

$$\begin{aligned} \text{L.H.S.} &= \frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} \\ &= \frac{[-\cos x][\cos x]}{(\sin x)(-\sin x)} \\ &= \frac{-\cos^2 x}{-\sin^2 x} \\ &= \cot^2 x \\ &= \text{R.H.S.} \end{aligned}$$

Question 9:

$$\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]=1$$

Answer 9:

$$\begin{aligned} \text{L.H.S.} &= \cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right] \\ &= \sin x \cos x [\tan x + \cot x] \\ &= \sin x \cos x \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) \\ &= (\sin x \cos x) \left[\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right] \\ &= 1 = \text{R.H.S.} \end{aligned}$$

Question 10:

Prove that $\sin (n + 1)x \sin (n + 2)x + \cos (n + 1)x \cos (n + 2)x = \cos x$

Answer 10:

L.H.S. = $\sin (n + 1)x \sin (n + 2)x + \cos (n + 1)x \cos (n + 2)x$

$$= \frac{1}{2} [2 \sin (n + 1)x \sin (n + 2)x + 2 \cos (n + 1)x \cos (n + 2)x]$$

$$= \frac{1}{2} \left[\cos \{ (n + 1)x - (n + 2)x \} - \cos \{ (n + 1)x + (n + 2)x \} \right. \\ \left. + \cos \{ (n + 1)x + (n + 2)x \} + \cos \{ (n + 1)x - (n + 2)x \} \right]$$

$$\left[\begin{array}{l} \because -2 \sin A \sin B = \cos (A + B) - \cos (A - B) \\ 2 \cos A \cos B = \cos (A + B) + \cos (A - B) \end{array} \right]$$

$$= \frac{1}{2} \times 2 \cos \{ (n + 1)x - (n + 2)x \}$$

$$= \cos (-x) = \cos x = \text{R.H.S.}$$

Question 11:

Prove that $\cos \left(\frac{3\pi}{4} + x \right) - \cos \left(\frac{3\pi}{4} - x \right) = -\sqrt{2} \sin x$

Answer 11:

It is known that $\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)$

$$\begin{aligned}\therefore \text{L.H.S.} &= \cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) \\ &= -2 \sin\left\{\frac{\left(\frac{3\pi}{4} + x\right) + \left(\frac{3\pi}{4} - x\right)}{2}\right\} \cdot \sin\left\{\frac{\left(\frac{3\pi}{4} + x\right) - \left(\frac{3\pi}{4} - x\right)}{2}\right\} \\ &= -2 \sin\left(\frac{3\pi}{4}\right) \sin x \\ &= -2 \sin\left(\pi - \frac{\pi}{4}\right) \sin x \\ &= -2 \sin \frac{\pi}{4} \sin x \\ &= -2 \times \frac{1}{\sqrt{2}} \times \sin x \\ &= -\sqrt{2} \sin x \\ &= \text{R.H.S.}\end{aligned}$$

Question 12:

Prove that $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

Answer 12:

It is known that

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right), \quad \sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\therefore \text{L.H.S.} = \sin^2 6x - \sin^2 4x$$

$$= (\sin 6x + \sin 4x) (\sin 6x - \sin 4x)$$

$$= \left[2 \sin\left(\frac{6x+4x}{2}\right) \cos\left(\frac{6x-4x}{2}\right) \right] \left[2 \cos\left(\frac{6x+4x}{2}\right) \sin\left(\frac{6x-4x}{2}\right) \right]$$

$$= (2 \sin 5x \cos x) (2 \cos 5x$$

$$\sin x) = (2 \sin 5x \cos 5x) (2$$

$$\sin x \cos x)$$

$$= \sin 10x \sin 2x$$

$$= \text{R.H.S.}$$

Question 13:

Prove that $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

Answer 13:

It is known that

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right), \quad \cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\therefore \text{L.H.S.} = \cos^2 2x - \cos^2 6x$$

$$= (\cos 2x + \cos 6x) (\cos 2x - \cos 6x)$$

$$= \left[2 \cos\left(\frac{2x+6x}{2}\right) \cos\left(\frac{2x-6x}{2}\right) \right] \left[-2 \sin\left(\frac{2x+6x}{2}\right) \sin\left(\frac{2x-6x}{2}\right) \right]$$

$$= [2 \cos 4x \cos(-2x)] [-2 \sin 4x \sin(-2x)]$$

$$= [2 \cos 4x \cos 2x] [-2 \sin 4x (-\sin 2x)]$$

$$\begin{aligned}
&= (2 \sin 4x \cos 4x) (2 \sin 2x \cos 2x) \\
&= \sin 8x \sin 4x = \text{R.H.S.}
\end{aligned}$$

Question 14:

Prove that $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

Answer 14:

$$\begin{aligned}
\text{L.H.S.} &= \sin 2x + 2 \sin 4x + \sin 6x \\
&= [\sin 2x + \sin 6x] + 2 \sin 4x \\
&= \left[2 \sin \left(\frac{2x+6x}{2} \right) \cos \left(\frac{2x-6x}{2} \right) \right] + 2 \sin 4x \\
&\left[\because \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\
&= 2 \sin 4x \cos (-2x) + 2 \sin 4x \\
&= 2 \sin 4x \cos 2x + 2 \sin 4x \\
&= 2 \sin 4x (\cos 2x + 1) \\
&= 2 \sin 4x (2 \cos^2 x - 1 + 1) \\
&= 2 \sin 4x (2 \cos^2 x) \\
&= 4\cos^2 x \sin 4x = \text{R.H.S.}
\end{aligned}$$

Question 15:

Prove that $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$

Answer 15:

$$\text{L.H.S} = \cot 4x (\sin 5x + \sin 3x)$$

$$= \frac{\cot 4x}{\sin 4x} \left[2 \sin \left(\frac{5x+3x}{2} \right) \cos \left(\frac{5x-3x}{2} \right) \right]$$

$$\left[\because \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right]$$

$$= \left(\frac{\cos 4x}{\sin 4x} \right) [2 \sin 4x \cos x]$$

$$= 2 \cos 4x \cos x$$

$$\text{R.H.S.} = \cot x (\sin 5x - \sin 3x)$$

$$= \frac{\cos x}{\sin x} \left[2 \cos \left(\frac{5x+3x}{2} \right) \sin \left(\frac{5x-3x}{2} \right) \right]$$

$$\left[\because \sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \right]$$

$$= \frac{\cos x}{\sin x} [2 \cos 4x \sin x]$$

$$= 2 \cos 4x \cdot \cos x$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Question 16:

Prove that $\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$

Answer 16:

It is known that

$$\cos A - \cos B = -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right), \quad \sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right)$$

$$\begin{aligned}
\therefore \text{L.H.S} &= \frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} \\
&= \frac{-2 \sin\left(\frac{9x+5x}{2}\right) \cdot \sin\left(\frac{9x-5x}{2}\right)}{2 \cos\left(\frac{17x+3x}{2}\right) \cdot \sin\left(\frac{17x-3x}{2}\right)} \\
&= \frac{-2 \sin 7x \cdot \sin 2x}{2 \cos 10x \cdot \sin 7x} \\
&= -\frac{\sin 2x}{\cos 10x} \\
&= \text{R.H.S.}
\end{aligned}$$

Question 17:

Prove that : $\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

Answer 17:

It is known that

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right), \quad \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\begin{aligned}
\therefore \text{L.H.S.} &= \frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} \\
&= \frac{2 \sin\left(\frac{5x+3x}{2}\right) \cdot \cos\left(\frac{5x-3x}{2}\right)}{2 \cos\left(\frac{5x+3x}{2}\right) \cdot \cos\left(\frac{5x-3x}{2}\right)} \\
&= \frac{2 \sin 4x \cdot \cos x}{2 \cos 4x \cdot \cos x} \\
&= \frac{\sin 4x}{\cos 4x} \\
&= \tan 4x = \text{R.H.S.}
\end{aligned}$$

Question 18:

Prove that $\frac{\sin x - \sin y}{\cos x + \cos y} = \tan \frac{x - y}{2}$

Answer 18:

It is known that

$$\sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right), \quad \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

$$\begin{aligned} \therefore \text{L.H.S.} &= \frac{\sin x - \sin y}{\cos x + \cos y} \\ &= \frac{2 \cos \left(\frac{x+y}{2} \right) \cdot \sin \left(\frac{x-y}{2} \right)}{2 \cos \left(\frac{x+y}{2} \right) \cdot \cos \left(\frac{x-y}{2} \right)} \\ &= \frac{\sin \left(\frac{x-y}{2} \right)}{\cos \left(\frac{x-y}{2} \right)} \\ &= \tan \left(\frac{x-y}{2} \right) = \text{R.H.S.} \end{aligned}$$

Question 19:

Prove that $\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

Answer 19:

It is known that

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right), \quad \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\begin{aligned} \therefore \text{L.H.S.} &= \frac{\sin x + \sin 3x}{\cos x + \cos 3x} \\ &= \frac{2 \sin\left(\frac{x+3x}{2}\right) \cos\left(\frac{x-3x}{2}\right)}{2 \cos\left(\frac{x+3x}{2}\right) \cos\left(\frac{x-3x}{2}\right)} \\ &= \frac{\sin 2x}{\cos 2x} \\ &= \tan 2x \\ &= \text{R.H.S.} \end{aligned}$$

Question 20:

Prove that $\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2 \sin x$

Answer 20:

It is known that

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right), \quad \cos^2 A - \sin^2 A = \cos 2A$$

$$\begin{aligned} \therefore \text{L.H.S.} &= \frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} \\ &= \frac{2 \cos\left(\frac{x+3x}{2}\right) \sin\left(\frac{x-3x}{2}\right)}{-\cos 2x} \\ &= \frac{2 \cos 2x \sin(-x)}{-\cos 2x} \\ &= -2 \times (-\sin x) \\ &= 2 \sin x = \text{R.H.S.} \end{aligned}$$

Question 21:

Prove that $\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

Answer 21:

$$\text{L.H.S.} = \frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$$

$$\begin{aligned}
&= \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x} \\
&= \frac{2 \cos \left(\frac{4x+2x}{2} \right) \cos \left(\frac{4x-2x}{2} \right) + \cos 3x}{2 \sin \left(\frac{4x+2x}{2} \right) \cos \left(\frac{4x-2x}{2} \right) + \sin 3x} \\
&\left[\because \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right), \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\
&= \frac{2 \cos 3x \cos x + \cos 3x}{2 \sin 3x \cos x + \sin 3x} \\
&= \frac{\cos 3x (2 \cos x + 1)}{\sin 3x (2 \cos x + 1)} \\
&= \cot 3x = \text{R.H.S.}
\end{aligned}$$

Question 22:

Prove that $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

Answer 22:

$$\begin{aligned}
\text{L.H.S.} &= \cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x \\
&= \cot x \cot 2x - \cot 3x (\cot 2x + \cot x) \\
&= \cot x \cot 2x - \cot (2x + x) (\cot 2x + \cot x)
\end{aligned}$$

$$\begin{aligned}
&= \cot x \cot 2x - \left[\frac{\cot 2x \cot x - 1}{\cot x + \cot 2x} \right] (\cot 2x + \cot x) \\
&\left[\because \cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B} \right]
\end{aligned}$$

$$= \cot x \cot 2x - (\cot 2x \cot x - 1) = 1 = \text{R.H.S.}$$

Question 23:

Prove that
$$\tan 4x = \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$$

Answer 23:

It is known that.
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

∴ L.H.S. = $\tan 4x = \tan 2(2x)$

$$\begin{aligned} &= \frac{2 \tan 2x}{1 - \tan^2 (2x)} \\ &= \frac{2 \left(\frac{2 \tan x}{1 - \tan^2 x} \right)}{1 - \left(\frac{2 \tan x}{1 - \tan^2 x} \right)^2} \\ &= \frac{\left(\frac{4 \tan x}{1 - \tan^2 x} \right)}{\left[1 - \frac{4 \tan^2 x}{(1 - \tan^2 x)^2} \right]} \\ &= \frac{\left(\frac{4 \tan x}{1 - \tan^2 x} \right)}{\left[\frac{(1 - \tan^2 x)^2 - 4 \tan^2 x}{(1 - \tan^2 x)^2} \right]} \\ &= \frac{4 \tan x (1 - \tan^2 x)}{(1 - \tan^2 x)^2 - 4 \tan^2 x} \\ &= \frac{4 \tan x (1 - \tan^2 x)}{1 + \tan^4 x - 2 \tan^2 x - 4 \tan^2 x} \\ &= \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x} = \text{R.H.S.} \end{aligned}$$

Question 24:

Prove that: $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

Answer 24:

$$\begin{aligned}\text{L.H.S.} &= \cos 4x \\ &= \cos 2(2x) \\ &= 1 - 2 \sin^2 2x \quad [\cos 2A = 1 - 2 \sin^2 A] \\ &= 1 - 2(2 \sin x \cos x)^2 \quad [\sin 2A = 2 \sin A \cos A] \\ &= 1 - 8 \sin^2 x \\ \cos^2 x &= \text{R.H.S.}\end{aligned}$$

Question 25:

Prove that: $\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$

Answer 25:

$$\begin{aligned}\text{L.H.S.} &= \cos 6x \\ &= \cos 3(2x) \\ &= 4 \cos^3 2x - 3 \cos 2x \quad [\cos 3A = 4 \cos^3 A - 3 \cos A] \\ &= 4 [(2 \cos^2 x - 1)^3 - 3 (2 \cos^2 x - 1) \cos 2x] \quad [\cos 2x = 2 \cos^2 x - 1] \\ &= 4 [(2 \cos^2 x)^3 - (1)^3 - 3 (2 \cos^2 x)^2 + 3 (2 \cos^2 x)] - 6 \cos^2 x + 3 \\ &= 4 [8 \cos^6 x - 1 - 12 \cos^4 x + 6 \cos^2 x] - 6 \cos^2 x + 3 \\ &= 32 \cos^6 x - 4 - 48 \cos^4 x + 24 \cos^2 x - 6 \cos^2 x + 3 \\ &= 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1 \\ \cos^2 x - 1 &= \text{R.H.S.}\end{aligned}$$

Exercise 3.4

Question 1:

Find the principal and general solutions of the equation $\tan x = \sqrt{3}$

Answer 1:

$$\tan x = \sqrt{3}$$

It is known that $\tan \frac{\pi}{3} = \sqrt{3}$ and $\tan \left(\frac{4\pi}{3} \right) = \tan \left(\pi + \frac{\pi}{3} \right) = \tan \frac{\pi}{3} = \sqrt{3}$

Therefore, the principal solutions are $x = \frac{\pi}{3}$ and $\frac{4\pi}{3}$.

$$\text{Now, } \tan x = \tan \frac{\pi}{3}$$

$$\Rightarrow x = n\pi + \frac{\pi}{3}, \text{ where } n \in \mathbb{Z}$$

Therefore, the general solution is $x = n\pi + \frac{\pi}{3}$, where $n \in \mathbb{Z}$

Question 2:

Find the principal and general solutions of the equation $\sec x = 2$

Answer 2:

$$\sec x = 2$$

It is known that $\sec \frac{\pi}{3} = 2$ and $\sec \frac{5\pi}{3} = \sec \left(2\pi - \frac{\pi}{3} \right) = \sec \frac{\pi}{3} = 2$

Therefore, the principal solutions are $x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.

$$\text{Now, } \sec x = \sec \frac{\pi}{3}$$

$$\Rightarrow \cos x = \cos \frac{\pi}{3} \quad \left[\sec x = \frac{1}{\cos x} \right]$$

$$\Rightarrow x = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in \mathbb{Z}$$

Therefore, the general solution is $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$

Question 3:

Find the principal and general solutions of the equation $\cot x = -\sqrt{3}$

Answer 3:

$$\cot x = -\sqrt{3}$$

It is known that $\cot \frac{\pi}{6} = \sqrt{3}$

$$\therefore \cot\left(\pi - \frac{\pi}{6}\right) = -\cot \frac{\pi}{6} = -\sqrt{3} \text{ and } \cot\left(2\pi - \frac{\pi}{6}\right) = -\cot \frac{\pi}{6} = -\sqrt{3}$$

$$\text{i.e., } \cot \frac{5\pi}{6} = -\sqrt{3} \text{ and } \cot \frac{11\pi}{6} = -\sqrt{3}$$

Therefore, the principal solutions are $x = \frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

$$\text{Now, } \cot x = \cot \frac{5\pi}{6}$$

$$\Rightarrow \tan x = \tan \frac{5\pi}{6} \quad \left[\cot x = \frac{1}{\tan x} \right]$$

$$\Rightarrow x = n\pi + \frac{5\pi}{6}, \text{ where } n \in \mathbf{Z}$$

Therefore, the general solution is $x = n\pi + \frac{5\pi}{6}$, where $n \in \mathbf{Z}$

Question 4:

Find the general solution of $\operatorname{cosec} x = -2$

Answer 4:

$$\operatorname{cosec} x = -2$$

It is known that

$$\operatorname{cosec} \frac{\pi}{6} = 2$$

$$\therefore \operatorname{cosec} \left(\pi + \frac{\pi}{6} \right) = -\operatorname{cosec} \frac{\pi}{6} = -2 \text{ and } \operatorname{cosec} \left(2\pi - \frac{\pi}{6} \right) = -\operatorname{cosec} \frac{\pi}{6} = -2$$

$$\text{i.e., } \operatorname{cosec} \frac{7\pi}{6} = -2 \text{ and } \operatorname{cosec} \frac{11\pi}{6} = -2$$

Therefore, the principal solutions are $x = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$$\text{Now, } \operatorname{cosec} x = \operatorname{cosec} \frac{7\pi}{6}$$

$$\Rightarrow \sin x = \sin \frac{7\pi}{6} \quad \left[\operatorname{cosec} x = \frac{1}{\sin x} \right]$$

$$\Rightarrow x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in \mathbb{Z}$$

Therefore, the general solution is $x = n\pi + (-1)^n \frac{7\pi}{6}$, where $n \in \mathbb{Z}$

Question 5:

Find the general solution of the equation $\cos 4x = \cos 2x$

Answer 5:

$$\cos 4x = \cos 2x$$

$$\Rightarrow \cos 4x - \cos 2x = 0$$

$$\Rightarrow -2 \sin\left(\frac{4x+2x}{2}\right) \sin\left(\frac{4x-2x}{2}\right) = 0$$

$$\left[\because \cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \right]$$

$$\Rightarrow \sin 3x \sin x = 0$$

$$\Rightarrow \sin 3x = 0 \quad \text{or} \quad \sin x = 0$$

$$\therefore 3x = n\pi \quad \text{or} \quad x = n\pi, \text{ where } n \in Z$$

$$\Rightarrow x = \frac{n\pi}{3} \quad \text{or} \quad x = n\pi, \text{ where } n \in Z$$

Question 6:

Find the general solution of the equation $\cos 3x + \cos x - \cos 2x = 0$

Answer 6:

$$\cos 3x + \cos x - \cos 2x = 0$$

$$\Rightarrow 2 \cos\left(\frac{3x+x}{2}\right) \cos\left(\frac{3x-x}{2}\right) - \cos 2x = 0 \quad \left[\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \right]$$

$$\Rightarrow 2 \cos 2x \cos x - \cos 2x = 0$$

$$\Rightarrow \cos 2x (2 \cos x - 1) = 0$$

$$\Rightarrow \cos 2x = 0 \quad \text{or} \quad 2 \cos x - 1 = 0$$

$$\Rightarrow \cos 2x = 0 \quad \text{or} \quad \cos x = \frac{1}{2}$$

$$\therefore 2x = (2n+1) \frac{\pi}{2} \quad \text{or} \quad \cos x = \cos \frac{\pi}{3}, \text{ where } n \in Z$$

$$\Rightarrow x = (2n+1) \frac{\pi}{4} \quad \text{or} \quad x = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in Z$$

Question 7:

Find the general solution of the equation $\sin 2x + \cos x = 0$

Answer 7:

$$\sin 2x + \cos x = 0$$

$$\Rightarrow 2 \sin x \cos x + \cos x = 0$$

$$\Rightarrow \cos x (2 \sin x + 1) = 0$$

$$\Rightarrow \cos x = 0 \quad \text{or} \quad 2 \sin x + 1 = 0$$

$$\text{Now, } \cos x = 0 \Rightarrow \cos x = (2n+1) \frac{\pi}{2}, \text{ where } n \in Z$$

$$2 \sin x + 1 = 0$$

$$\Rightarrow \sin x = \frac{-1}{2} = -\sin \frac{\pi}{6} = \sin \left(\pi + \frac{\pi}{6} \right) = \sin \left(\pi + \frac{\pi}{6} \right) = \sin \frac{7\pi}{6}$$

$$\Rightarrow x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in Z$$

Therefore, the general solution is $(2n+1) \frac{\pi}{2}$ or $n\pi + (-1)^n \frac{7\pi}{6}$, $n \in Z$

Question 8:

Find the general solution of the equation $\sec^2 2x = 1 - \tan 2x$

Answer 8:

$$\sec^2 2x = 1 - \tan 2x$$

$$\Rightarrow 1 + \tan^2 2x = 1 - \tan 2x$$

$$\Rightarrow \tan^2 2x + \tan 2x = 0$$

$$\Rightarrow \tan 2x (\tan 2x + 1) = 0$$

$$\Rightarrow \tan 2x = 0 \quad \text{or} \quad \tan 2x + 1 = 0$$

Now, $\tan 2x = 0$

$$\Rightarrow \tan 2x = \tan 0$$

$$\Rightarrow 2x = n\pi + 0, \text{ where } n \in Z$$

$$\Rightarrow x = \frac{n\pi}{2}, \text{ where } n \in Z$$

$$\tan 2x + 1 = 0$$

$$\Rightarrow \tan 2x = -1 = -\tan \frac{\pi}{4} = \tan \left(\pi - \frac{\pi}{4} \right) = \tan \frac{3\pi}{4}$$

$$\Rightarrow 2x = n\pi + \frac{3\pi}{4}, \text{ where } n \in Z$$

$$\Rightarrow x = \frac{n\pi}{2} + \frac{3\pi}{8}, \text{ where } n \in Z$$

Therefore, the general solution is $\frac{n\pi}{2}$ or $\frac{n\pi}{2} + \frac{3\pi}{8}$, $n \in Z$

Question 9:

Find the general solution of the equation $\sin x + \sin 3x + \sin 5x = 0$

Answer 9:

$$\sin x + \sin 3x + \sin 5x = 0$$

$$(\sin x + \sin 5x) + \sin 3x = 0$$

$$\Rightarrow \left[2 \sin \left(\frac{x+5x}{2} \right) \cos \left(\frac{x-5x}{2} \right) \right] + \sin 3x = 0$$

$$\left[\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right]$$

$$\Rightarrow 2 \sin 3x \cos(-2x) + \sin 3x = 0$$

$$\Rightarrow 2 \sin 3x \cos 2x + \sin 3x = 0$$

$$\Rightarrow \sin 3x (2 \cos 2x + 1) = 0$$

$$\Rightarrow \sin 3x = 0 \quad \text{or} \quad 2 \cos 2x + 1 = 0$$

Now, $\sin 3x = 0 \Rightarrow 3x = n\pi$, where $n \in Z$

i.e., $x = \frac{n\pi}{3}$, where $n \in Z$

$$2 \cos 2x + 1 = 0$$

$$\Rightarrow \cos 2x = \frac{-1}{2} = -\cos \frac{\pi}{3} = \cos \left(\pi - \frac{\pi}{3} \right)$$

$$\Rightarrow \cos 2x = \cos \frac{2\pi}{3}$$

$$\Rightarrow 2x = 2n\pi \pm \frac{2\pi}{3}, \text{ where } n \in Z$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{3}, \text{ where } n \in Z$$

Therefore, the general solution is $\frac{n\pi}{3}$ or $n\pi \pm \frac{\pi}{3}$, $n \in Z$

Mathematics

(Chapter – 3) (Trigonometric Functions)

(Class – XI)

Miscellaneous Exercise on chapter 3

Question 1:

Prove that: $2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0$

Answer 1:

L.H.S.

$$\begin{aligned} &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \left(\frac{\frac{3\pi}{13} + \frac{5\pi}{13}}{2} \right) \cos \left(\frac{\frac{3\pi}{13} - \frac{5\pi}{13}}{2} \right) \left[\cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) \right] \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \left(\frac{-\pi}{13} \right) \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \frac{\pi}{13} \\ &= 2 \cos \frac{\pi}{13} \left[\cos \frac{9\pi}{13} + \cos \frac{4\pi}{13} \right] \\ &= 2 \cos \frac{\pi}{13} \left[2 \cos \left(\frac{\frac{9\pi}{13} + \frac{4\pi}{13}}{2} \right) \cos \left(\frac{\frac{9\pi}{13} - \frac{4\pi}{13}}{2} \right) \right] \\ &= 2 \cos \frac{\pi}{13} \left[2 \cos \frac{\pi}{2} \cos \frac{5\pi}{26} \right] \\ &= 2 \cos \frac{\pi}{13} \times 2 \times 0 \times \cos \frac{5\pi}{26} \end{aligned}$$

= 0 = R.H.S

Question 2:

Prove that: $(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0$

Answer 2:

L.H.S.

$$\begin{aligned}
 &= (\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x \\
 &= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x - \cos^2 x \\
 &= \cos 3x \cos x + \sin 3x \sin x - (\cos^2 x - \sin^2 x) \\
 &= \cos(3x - x) - \cos 2x \quad \left[\cos(A - B) = \cos A \cos B + \sin A \sin B \right] \\
 &= \cos 2x - \cos 2x \\
 &= 0 \\
 &= \text{R.H.S.}
 \end{aligned}$$

Question 3:

Prove that: $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4 \cos^2 \frac{x+y}{2}$

Answer 3:

$$\begin{aligned}
 \text{L.H.S.} &= (\cos x + \cos y)^2 + (\sin x - \sin y)^2 \\
 &= \cos^2 x + \cos^2 y + 2 \cos x \cos y + \sin^2 x + \sin^2 y - 2 \sin x \sin y \\
 &= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) + 2(\cos x \cos y - \sin x \sin y) \\
 &= 1 + 1 + 2 \cos(x + y) \quad \left[\cos(A + B) = (\cos A \cos B - \sin A \sin B) \right] \\
 &= 2 + 2 \cos(x + y) \\
 &= 2 \left[1 + \cos(x + y) \right] \\
 &= 2 \left[1 + 2 \cos^2 \left(\frac{x+y}{2} \right) - 1 \right] \quad \left[\cos 2A = 2 \cos^2 A - 1 \right] \\
 &= 4 \cos^2 \left(\frac{x+y}{2} \right) = \text{R.H.S.}
 \end{aligned}$$

Question 4:

Prove that: $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \frac{x-y}{2}$

Answer 4:

L.H.S.

$$\begin{aligned} &= (\cos x - \cos y)^2 + (\sin x - \sin y)^2 \\ &= \cos^2 x + \cos^2 y - 2 \cos x \cos y + \sin^2 x + \sin^2 y - 2 \sin x \sin y \\ &= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2[\cos x \cos y + \sin x \sin y] \\ &= 1 + 1 - 2[\cos(x-y)] \quad [\cos(A-B) = \cos A \cos B + \sin A \sin B] \\ &= 2[1 - \cos(x-y)] \\ &= 2 \left[1 - \left\{ 1 - 2 \sin^2 \left(\frac{x-y}{2} \right) \right\} \right] \quad [\cos 2A = 1 - 2 \sin^2 A] \\ &= 4 \sin^2 \left(\frac{x-y}{2} \right) = \text{R.H.S.} \end{aligned}$$

Question 5:

Prove that: $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \cos x \cos 2x \sin 4x$

Answer 5:

$$\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cdot \cos \left(\frac{A-B}{2} \right)$$

It is known that

$$\square \text{L.H.S.} = \sin x + \sin 3x + \sin 5x + \sin 7x$$

$$= (\sin x + \sin 5x) + (\sin 3x + \sin 7x)$$

$$= 2 \sin \left(\frac{x+5x}{2} \right) \cdot \cos \left(\frac{x-5x}{2} \right) + 2 \sin \left(\frac{3x+7x}{2} \right) \cos \left(\frac{3x-7x}{2} \right)$$

$$= 2 \sin 3x \cos(-2x) + 2 \sin 5x \cos(-2x)$$

$$= 2 \sin 3x \cos 2x + 2 \sin 5x \cos 2x$$

$$= 2 \cos 2x [\sin 3x + \sin 5x]$$

$$= 2 \cos 2x \left[2 \sin \left(\frac{3x+5x}{2} \right) \cdot \cos \left(\frac{3x-5x}{2} \right) \right]$$

$$= 2 \cos 2x [2 \sin 4x \cdot \cos(-x)]$$

$$= 4 \cos 2x \sin 4x \cos x = \text{R.H.S.}$$

Question 6:

Prove that:
$$\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$$

Answer 6:

It is known that

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right), \quad \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)$$

$$\text{L.H.S.} = \frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)}$$

$$\begin{aligned} &= \frac{\left[2 \sin\left(\frac{7x+5x}{2}\right) \cdot \cos\left(\frac{7x-5x}{2}\right)\right] + \left[2 \sin\left(\frac{9x+3x}{2}\right) \cdot \cos\left(\frac{9x-3x}{2}\right)\right]}{\left[2 \cos\left(\frac{7x+5x}{2}\right) \cdot \cos\left(\frac{7x-5x}{2}\right)\right] + \left[2 \cos\left(\frac{9x+3x}{2}\right) \cdot \cos\left(\frac{9x-3x}{2}\right)\right]} \\ &= \frac{[2 \sin 6x \cdot \cos x] + [2 \sin 6x \cdot \cos 3x]}{[2 \cos 6x \cdot \cos x] + [2 \cos 6x \cdot \cos 3x]} \\ &= \frac{2 \sin 6x [\cos x + \cos 3x]}{2 \cos 6x [\cos x + \cos 3x]} \end{aligned}$$

$$= \tan 6x$$

$$= \text{R.H.S.}$$

Question 7:

Prove that: $\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$

Answer 7:

$$\text{L.H.S.} = \sin 3x + \sin 2x - \sin x$$

$$= \sin 3x + (\sin 2x - \sin x)$$

$$= \sin 3x + \left[2 \cos \left(\frac{2x+x}{2} \right) \sin \left(\frac{2x-x}{2} \right) \right] \quad \left[\sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \right]$$

$$= \sin 3x + \left[2 \cos \left(\frac{3x}{2} \right) \sin \left(\frac{x}{2} \right) \right]$$

$$= \sin 3x + 2 \cos \frac{3x}{2} \sin \frac{x}{2}$$

$$= 2 \sin \frac{3x}{2} \cdot \cos \frac{3x}{2} + 2 \cos \frac{3x}{2} \sin \frac{x}{2} \quad [\sin 2A = 2 \sin A \cdot \cos B]$$

$$= 2 \cos \left(\frac{3x}{2} \right) \left[\sin \left(\frac{3x}{2} \right) + \sin \left(\frac{x}{2} \right) \right]$$

$$= 2 \cos \left(\frac{3x}{2} \right) \left[2 \sin \left\{ \frac{\left(\frac{3x}{2} \right) + \left(\frac{x}{2} \right)}{2} \right\} \cos \left\{ \frac{\left(\frac{3x}{2} \right) - \left(\frac{x}{2} \right)}{2} \right\} \right] \quad \left[\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right]$$

$$= 2 \cos \left(\frac{3x}{2} \right) \cdot 2 \sin x \cos \left(\frac{x}{2} \right)$$

$$= 4 \sin x \cos \left(\frac{x}{2} \right) \cos \left(\frac{3x}{2} \right) = \text{R.H.S.}$$

Question 8:

Find $\sin x/2$, $\cos x/2$ and $\tan x/2$, if $\tan x = -\frac{4}{3}$, x in quadrant II

Answer 8:

Here, x is in quadrant II.

$$\text{i.e., } \frac{\pi}{2} < x < \pi$$

$$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Therefore, $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$

are lies in first quadrant.

It is given that $\tan x = -\frac{4}{3}$.

$$\sec^2 x = 1 + \tan^2 x = 1 + \left(-\frac{4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9}$$

$$\therefore \cos^2 x = \frac{9}{25}$$

$$\Rightarrow \cos x = \pm \frac{3}{5}$$

As x is in quadrant II, $\cos x$ is negative.

$$\square \quad \cos x = \frac{-3}{5}$$

$$\text{Now, } \cos x = 2 \cos^2 \frac{x}{2} - 1$$

$$\Rightarrow \frac{-3}{5} = 2 \cos^2 \frac{x}{2} - 1$$

$$\Rightarrow 2 \cos^2 \frac{x}{2} = 1 - \frac{3}{5}$$

$$\Rightarrow 2 \cos^2 \frac{x}{2} = \frac{2}{5}$$

$$\Rightarrow \cos^2 \frac{x}{2} = \frac{1}{5}$$

$$\Rightarrow \cos \frac{x}{2} = \frac{1}{\sqrt{5}} \quad \left[\because \cos \frac{x}{2} \text{ is positive} \right]$$

$$\therefore \cos \frac{x}{2} = \frac{\sqrt{5}}{5}$$

$$\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1$$

$$\Rightarrow \sin^2 \frac{x}{2} + \left(\frac{1}{\sqrt{5}} \right)^2 = 1$$

$$\Rightarrow \sin^2 \frac{x}{2} = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\Rightarrow \sin \frac{x}{2} = \frac{2}{\sqrt{5}} \quad \left[\because \sin \frac{x}{2} \text{ is positive} \right]$$

$$\therefore \sin \frac{x}{2} = \frac{2\sqrt{5}}{5}$$

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{2}{\sqrt{5}} \right)}{\left(\frac{1}{\sqrt{5}} \right)} = 2$$

Thus, the respective values of $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are $\frac{2\sqrt{5}}{5}$, $\frac{\sqrt{5}}{5}$, and 2.

Question 9:

Find , $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ for $\cos x = -\frac{1}{3}$, x in quadrant III

Answer 9:

Here, x is in quadrant III.

$$\text{i.e., } \pi < x < \frac{3\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$

Therefore, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are negative, where $\sin \frac{x}{2}$ as is positive.

It is given that $\cos x = -\frac{1}{3}$.

$$\cos x = 1 - 2\sin^2 \frac{x}{2}$$

$$\Rightarrow \sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$$

$$\Rightarrow \sin^2 \frac{x}{2} = \frac{1 - \left(-\frac{1}{3}\right)}{2} = \frac{\left(1 + \frac{1}{3}\right)}{2} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

$$\Rightarrow \sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \quad \left[\because \sin \frac{x}{2} \text{ is positive} \right]$$

$$\therefore \sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

Now

$$\cos x = 2\cos^2 \frac{x}{2} - 1$$

$$\Rightarrow \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} = \frac{1 + \left(-\frac{1}{3}\right)}{2} = \frac{\left(\frac{3-1}{3}\right)}{2} = \frac{\left(\frac{2}{3}\right)}{2} = \frac{1}{3}$$

$$\Rightarrow \cos \frac{x}{2} = -\frac{1}{\sqrt{3}} \quad \left[\because \cos \frac{x}{2} \text{ is negative} \right]$$

$$\therefore \cos \frac{x}{2} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$$

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{\left(\frac{-1}{\sqrt{3}}\right)} = -\sqrt{2}$$

Thus, the respective values of $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are $\frac{\sqrt{6}}{3}$, $\frac{-\sqrt{3}}{3}$, and $-\sqrt{2}$.

Question 10:

Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ for $\sin x = \frac{1}{4}$, x in quadrant II

Answer 10:

Here, x is in quadrant II.

$$\text{i.e., } \frac{\pi}{2} < x < \pi$$

$$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Therefore, $\sin \frac{x}{2}$, $\cos \frac{x}{2}$, $\tan \frac{x}{2}$ are all positive.

It is given that $\sin x = \frac{1}{4}$.

$$\cos^2 x = 1 - \sin^2 x = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\Rightarrow \cos x = -\frac{\sqrt{15}}{4} \quad [\cos x \text{ is negative in quadrant II}]$$

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} = \frac{1 - \left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4 + \sqrt{15}}{8}$$

$$\Rightarrow \sin \frac{x}{2} = \sqrt{\frac{4 + \sqrt{15}}{8}} \quad \left[\because \sin \frac{x}{2} \text{ is positive} \right]$$

$$= \sqrt{\frac{4 + \sqrt{15}}{8} \times \frac{2}{2}}$$

$$= \sqrt{\frac{8 + 2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8 + 2\sqrt{15}}}{4}$$

$$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} = \frac{1 + \left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4 - \sqrt{15}}{8}$$

$$\Rightarrow \cos \frac{x}{2} = \sqrt{\frac{4 - \sqrt{15}}{8}} \quad \left[\because \cos \frac{x}{2} \text{ is positive} \right]$$

$$= \sqrt{\frac{4 - \sqrt{15}}{8} \times \frac{2}{2}}$$

$$= \sqrt{\frac{8 - 2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8 - 2\sqrt{15}}}{4}$$

$$\begin{aligned}
 \tan \frac{x}{2} &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4} \right)}{\left(\frac{\sqrt{8-2\sqrt{15}}}{4} \right)} = \frac{\sqrt{8+2\sqrt{15}}}{\sqrt{8-2\sqrt{15}}} \\
 &= \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}} \times \frac{8+2\sqrt{15}}{8+2\sqrt{15}}} \\
 &= \sqrt{\frac{(8+2\sqrt{15})^2}{64-60}} = \frac{8+2\sqrt{15}}{2} = 4+\sqrt{15}
 \end{aligned}$$

Thus, the respective values of $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$

are $\frac{\sqrt{8+2\sqrt{15}}}{4}$, $\frac{\sqrt{8-2\sqrt{15}}}{4}$,

and $4+\sqrt{15}$

Mathematics

(Chapter - 3) (Trigonometric Functions) (Supplementary Exercise)
(Class 11)

Exercise 3.5

Question 1:

In any triangle ABC, if $a = 18, b = 24, c = 30$, find: $\cos A, \cos B, \cos C$.

Answer 1:

Using cosine formula $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, we have

$$\cos A = \frac{24^2 + 30^2 - 18^2}{2 \times 24 \times 30} = \frac{576 + 900 - 324}{1440} = \frac{1152}{1440} = \frac{4}{5}$$

Similarly, using $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$, we have

$$\cos B = \frac{30^2 + 18^2 - 24^2}{2 \times 30 \times 18} = \frac{900 + 324 - 576}{1080} = \frac{648}{1080} = \frac{3}{5}$$

and using $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, we get

$$\cos C = \frac{18^2 + 24^2 - 30^2}{2 \times 18 \times 24} = \frac{324 + 576 - 900}{864} = \frac{0}{864} = 0$$

Question 2:

In any triangle ABC, if $a = 18, b = 24, c = 30$, find: $\cos A, \cos B, \cos C$.

Answer 2:

Using cosine formula $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, we have

$$\cos A = \frac{24^2 + 30^2 - 18^2}{2 \times 24 \times 30} = \frac{576 + 900 - 324}{1440} = \frac{1152}{1440} = \frac{4}{5}$$

$$\sin A = \sqrt{1 - \cos^2 A} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

Similarly, using $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$, we have

$$\cos B = \frac{30^2 + 18^2 - 24^2}{2 \times 30 \times 18} = \frac{900 + 324 - 576}{1080} = \frac{648}{1080} = \frac{3}{5}$$

$$\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

and using $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, we get

$$\cos C = \frac{18^2 + 24^2 - 30^2}{2 \times 18 \times 24} = \frac{324 + 576 - 900}{864} = \frac{0}{864} = 0$$

$$\sin C = \sqrt{1 - \cos^2 C} = \sqrt{1 - (0)^2} = \sqrt{1 - 0} = \sqrt{1} = 1$$

Question 3:

For any triangle ABC, prove that: $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$

Answer 3:

$$\begin{aligned}
\text{LHS} &= \frac{a+b}{c} \\
&= \frac{k \sin A + k \sin B}{k \sin C} && \left[\because \text{Using } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \right] \\
&= \frac{k(\sin A + \sin B)}{k \sin C} = \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{\sin C} && \left[\because \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \right] \\
&= \frac{2 \sin \left(90^\circ - \frac{C}{2}\right) \cos \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}} && \left[\because \frac{A+B}{2} = 90^\circ - \frac{C}{2} \text{ and } \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right] \\
&= \frac{\cos \frac{C}{2} \cos \frac{A-B}{2}}{\sin \frac{C}{2} \cos \frac{C}{2}} = \frac{\cos \left(\frac{A-B}{2}\right)}{\sin \frac{C}{2}} = \text{RHS}
\end{aligned}$$

Question 4:

For any triangle ABC, prove that: $\frac{a-b}{c} = \frac{\sin \left(\frac{A-B}{2}\right)}{\cos \frac{C}{2}}$

Answer 4:

$$\begin{aligned}
\text{LHS} &= \frac{a-b}{c} \\
&= \frac{k \sin A - k \sin B}{k \sin C} && \left[\because \text{Using } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \right] \\
&= \frac{k(\sin A - \sin B)}{k \sin C} = \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{\sin C} && \left[\because \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \right] \\
&= \frac{2 \cos \left(90^\circ - \frac{C}{2}\right) \sin \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}} && \left[\because \frac{A+B}{2} = 90^\circ - \frac{C}{2} \text{ and } \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right] \\
&= \frac{\sin \frac{C}{2} \sin \frac{A-B}{2}}{\sin \frac{C}{2} \cos \frac{C}{2}} = \frac{\sin \left(\frac{A-B}{2}\right)}{\cos \frac{C}{2}} = \text{RHS}
\end{aligned}$$

Question 5:

For any triangle ABC, prove that: $\sin \frac{B-C}{2} = \frac{b-c}{a} \cos \frac{A}{2}$

Answer 5:

$$\begin{aligned}
\text{RHS} &= \frac{b-c}{a} \cos \frac{A}{2} \\
&= \frac{k \sin B - k \sin C}{k \sin A} \cos \frac{A}{2} && \left[\because \text{Using } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \right] \\
&= \frac{k(\sin B - \sin C)}{k \sin A} \cos \frac{A}{2} = \frac{2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}}{\sin A} \cos \frac{A}{2} && \left[\because \sin B - \sin C = 2 \cos \frac{B+C}{2} \sin \frac{B-C}{2} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{2 \cos\left(90 - \frac{A}{2}\right) \sin \frac{B-C}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} \cos \frac{A}{2} & \left[\because \frac{A+B}{2} = 90^\circ - \frac{C}{2} \text{ and } \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right] \\
&= \frac{\sin \frac{A}{2} \sin \frac{B-C}{2}}{\sin \frac{A}{2}} = \sin \frac{B-C}{2} = \text{LHS}
\end{aligned}$$

Question 6:

For any triangle ABC, prove that: $a(b \cos C - c \cos B) = b^2 - c^2$

Answer 6:

$$\begin{aligned}
\text{LHS} &= a(b \cos C - c \cos B) \\
&= a \left[b \left(\frac{a^2 + b^2 - c^2}{2ab} \right) - c \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \right] & \left[\because \text{Using } \cos A = \frac{b^2 + c^2 - a^2}{2bc} \right] \\
&= a \left[\frac{a^2 + b^2 - c^2}{2a} - \frac{a^2 + c^2 - b^2}{2a} \right] = a \left[\frac{a^2 + b^2 - c^2 - a^2 - c^2 + b^2}{2a} \right] \\
&= a \left[\frac{2(b^2 - c^2)}{2a} \right] = b^2 - c^2 = \text{RHS}
\end{aligned}$$

Question 7:

For any triangle ABC, prove that: $a(\cos C - \cos B) = 2(b-c) \cos^2 \frac{A}{2}$

Answer 7:

$$\begin{aligned}
\text{LHS} &= a(\cos C - \cos B) \\
&= a \left[\left(\frac{a^2 + b^2 - c^2}{2ab} \right) - \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \right] & \left[\because \text{Using } \cos A = \frac{b^2 + c^2 - a^2}{2bc} \right] \\
&= a \left[\frac{ca^2 + cb^2 - c^3 - ba^2 - bc^2 + b^3}{2abc} \right] \\
&= \frac{b^3 - c^3 + b^2c - bc^2 + a^2c - a^2b}{2bc} = \frac{(b-c)(b^2 + bc + c^2) + bc(b-c) - a^2(b-c)}{2bc} \\
&= (b-c) \left[\frac{b^2 + bc + c^2 + bc - a^2}{2bc} \right] = (b-c) \left[\frac{2bc + b^2 + c^2 - a^2}{2bc} \right] \\
&= (b-c) [1 + \cos A] & \left[\because \text{Using } \cos A = \frac{b^2 + c^2 - a^2}{2bc} \right] \\
&= 2(b-c) \cos^2 \frac{A}{2} = \text{RHS} & \left[\because 1 + \cos A = 2 \cos^2 \frac{A}{2} \right]
\end{aligned}$$

Question 8:

For any triangle ABC, prove that: $\frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2 - c^2}{a^2}$

Answer 8:

$$\begin{aligned}
\text{LHS} &= \frac{\sin(B-C)}{\sin(B+C)} = \frac{\sin B \cos C - \cos B \sin C}{\sin B \cos C + \cos B \sin C} \\
&= \frac{kb \left(\frac{a^2 + b^2 - c^2}{2ab} \right) - \left(\frac{a^2 + c^2 - b^2}{2ac} \right) kc}{kb \left(\frac{a^2 + b^2 - c^2}{2ab} \right) + \left(\frac{a^2 + c^2 - b^2}{2ac} \right) kc} & \left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ and } \cos A = \frac{b^2 + c^2 - a^2}{2bc} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\left(\frac{a^2 + b^2 - c^2}{2a} - \frac{a^2 + c^2 - b^2}{2a}\right)}{\left(\frac{a^2 + b^2 - c^2}{2a} + \frac{a^2 + c^2 - b^2}{2ac}\right)} = \frac{\left(\frac{a^2 + b^2 - c^2 - a^2 - c^2 + b^2}{2a}\right)}{\left(\frac{a^2 + b^2 - c^2 + a^2 + c^2 - b^2}{2a}\right)} = \frac{2(b^2 - c^2)}{2(a^2)} \\
&= \frac{b^2 - c^2}{a^2} = \text{RHS}
\end{aligned}$$

Question 9:

For any triangle ABC, prove that: $(b + c) \cos \frac{B+C}{2} = a \cos \frac{B-C}{2}$

Answer 9:

$$\begin{aligned}
\text{LHS} &= (b + c) \cos \frac{B+C}{2} \\
&= (k \sin B + k \sin C) \cos \frac{B+C}{2} && \left[\because \text{Using } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \right] \\
&= k(\sin B + \sin C) \cos \frac{B+C}{2} \\
&= k \left(2 \sin \frac{B+C}{2} \cos \frac{B-C}{2} \right) \cos \frac{B+C}{2} && \left[\because \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \right] \\
&= 2k \sin \left(90 - \frac{A}{2} \right) \cos \frac{B-C}{2} \cos \left(90 - \frac{A}{2} \right) && \left[\because \frac{A+B}{2} = 90^\circ - \frac{C}{2} \right] \\
&= 2k \cos \frac{A}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \\
&= k \sin A \cos \frac{B-C}{2} && \left[\because 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \sin \theta \right] \\
&= a \cos \frac{B-C}{2} = \text{RHS}
\end{aligned}$$

Question 10:

For any triangle ABC, prove that: $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$

Answer 10:

$$\begin{aligned}
\text{LHS} &= a \cos A + b \cos B + c \cos C \\
&= k \sin A \cos A + k \sin B \cos B + k \sin C \cos C && \left[\because \text{Using } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \right] \\
&= \frac{k}{2} (2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C) \\
&= \frac{k}{2} (\sin 2A + \sin 2B + 2 \sin C \cos C) \\
&= \frac{k}{2} [2 \sin(A+B) \cos(A-B) + 2 \sin C \cos C] && \left[\because \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \right] \\
&= \frac{k}{2} [2 \sin(180 - C) \cos(A-B) + 2 \sin C \cos C] && \left[\because A+B = 180^\circ - C \right] \\
&= \frac{k}{2} [2 \sin C \cos(A-B) + 2 \sin C \cos C] = k \sin C [\cos(A-B) + \cos C] \\
&= k \sin C [\cos(A-B) + \cos(180 - (A+B))] && \left[\because A+B+C = 180^\circ \right] \\
&= k \sin C [\cos(A-B) - \cos(A+B)] \\
&= k \sin C [2 \sin A \sin B] && \left[\because \cos(A-B) - \cos(A+B) = 2 \sin A \sin B \right] \\
&= 2a \sin B \sin C = \text{RHS} && \left[\because k \sin A = a \right]
\end{aligned}$$

Question 11:

For any triangle ABC, prove that: $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$

Answer 11:

$$\begin{aligned} \text{LHS} &= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} \\ &= \frac{1}{a} \left(\frac{b^2 + c^2 - a^2}{2bc} \right) + \frac{1}{b} \left(\frac{a^2 + c^2 - b^2}{2ac} \right) + \frac{1}{c} \left(\frac{a^2 + b^2 - c^2}{2ab} \right) \left[\because \text{Using } \cos A = \frac{b^2 + c^2 - a^2}{2bc} \right] \\ &= \frac{b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc} = \text{RHS} \end{aligned}$$

Question 12:

For any triangle ABC, prove that: $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

Answer 12:

$$\begin{aligned} \text{LHS} &= (b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C \\ &= (b^2 - c^2) \frac{\cos A}{\sin A} + (c^2 - a^2) \frac{\cos B}{\sin B} + (a^2 - b^2) \frac{\cos C}{\sin C} \\ &= (b^2 - c^2) \left[\frac{1}{ka} \left(\frac{b^2 + c^2 - a^2}{2bc} \right) \right] + (c^2 - a^2) \left[\frac{1}{kb} \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \right] + (a^2 - b^2) \left[\frac{1}{kc} \left(\frac{a^2 + b^2 - c^2}{2ab} \right) \right] \\ &\quad \left[\because \text{Using } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ and } \cos A = \frac{b^2 + c^2 - a^2}{2bc} \right] \\ &= \frac{1}{kabc} [(b^2 - c^2)(b^2 + c^2 - a^2) + (c^2 - a^2)(a^2 + c^2 - b^2) + (a^2 - b^2)(a^2 + b^2 - c^2)] \\ &= \frac{1}{kabc} [(b^4 + b^2c^2 - a^2b^2 - b^2c^2 - c^4 + c^2a^2) + (c^2a^2 + c^4 - b^2c^2 - a^4 - c^2a^2 + a^2b^2) + (a^4 + a^2b^2 - c^2a^2 - a^2b^2 - b^4 + b^2c^2)] \\ &= \frac{1}{kabc} (0) = 0 = \text{RHS} \end{aligned}$$

Question 13:

For any triangle ABC, prove that: $\frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0$

Answer 13:

$$\begin{aligned} \text{LHS} &= \frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C \\ &= \frac{b^2 - c^2}{a^2} 2 \sin A \cos A + \frac{c^2 - a^2}{b^2} 2 \sin B \cos B + \frac{a^2 - b^2}{c^2} 2 \sin C \cos C \quad [\because \sin 2A = 2 \sin A \cos A] \\ &= \left(\frac{b^2 - c^2}{a^2} \right) \left[ka \left(\frac{b^2 + c^2 - a^2}{2bc} \right) \right] + \left(\frac{c^2 - a^2}{b^2} \right) \left[kb \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \right] + \left(\frac{a^2 - b^2}{c^2} \right) \left[kc \left(\frac{a^2 + b^2 - c^2}{2ab} \right) \right] \\ &\quad \left[\because \text{Using } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ and } \cos A = \frac{b^2 + c^2 - a^2}{2bc} \right] \\ &= \frac{k}{abc} [(b^2 - c^2)(b^2 + c^2 - a^2) + (c^2 - a^2)(a^2 + c^2 - b^2) + (a^2 - b^2)(a^2 + b^2 - c^2)] \\ &= \frac{k}{abc} [(b^4 + b^2c^2 - a^2b^2 - b^2c^2 - c^4 + c^2a^2) + (c^2a^2 + c^4 - b^2c^2 - a^4 - c^2a^2 + a^2b^2) + (a^4 + a^2b^2 - c^2a^2 - a^2b^2 - b^4 + b^2c^2)] \\ &= \frac{k}{abc} (0) = 0 = \text{RHS} \end{aligned}$$