

CHAPTER - 5

COMPLEX NUMBERS & QUADRATIC EQUATIONS

Exercise 5.1

Question 1:

Express the given complex number in the form $a + ib$: $(5i)\left(-\frac{3}{5}i\right)$

Answer 1:

$$\begin{aligned}(5i)\left(-\frac{3}{5}i\right) &= -5 \times \frac{3}{5} \times i \times i \\ &= -3i^2 \\ &= -3(-1) \quad [i^2 = -1] \\ &= 3\end{aligned}$$

Question 2:

Express the given complex number in the form $a + ib$: $i^9 + i^{19}$

Answer 2:

$$\begin{aligned}i^9 + i^{19} &= i^{4 \times 2 + 1} + i^{4 \times 4 + 3} \\ &= (i^4)^2 \cdot i + (i^4)^4 \cdot i^3 \\ &= 1 \times i + 1 \times (-i) \quad [i^4 = 1, i^3 = -i] \\ &= i + (-i) \\ &= 0\end{aligned}$$

Question 3:

Express the given complex number in the form $a + ib$: i^{-39}

Answer 3:

$$\begin{aligned}
 i^{-39} &= i^{-4 \times 9 - 3} = (i^4)^{-9} \cdot i^{-3} \\
 &= (1)^{-9} \cdot i^{-3} \quad [i^4 = 1] \\
 &= \frac{1}{i^3} = \frac{1}{-i} \quad [i^3 = -i] \\
 &= \frac{-1}{i} \times \frac{i}{i} \\
 &= \frac{-i}{i^2} = \frac{-i}{-1} = i \quad [i^2 = -1]
 \end{aligned}$$

Question 4:

Express the given complex number in the form $a + ib$:

$$3(7 + i7) + i(7 + i7)$$

Answer 4:

$$\begin{aligned}
 3(7 + i7) + i(7 + i7) &= 21 + 21i + 7i + 7i^2 \\
 &= 21 + 28i + 7 \times (-1) \quad [\because i^2 = -1] \\
 &= 14 + 28i
 \end{aligned}$$

Question 5:

Express the given complex number in the form $a + ib$: $(1 - i) - (-1 + i6)$

Answer 5:

$$\begin{aligned}
 (1 - i) - (-1 + i6) &= 1 - i + 1 - 6i \\
 &= 2 - 7i
 \end{aligned}$$

Question 6:

Express the given complex number in the form $a + ib$: $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$

Answer 6:

$$\begin{aligned} & \left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right) \\ &= \frac{1}{5} + \frac{2}{5}i - 4 - \frac{5}{2}i \\ &= \left(\frac{1}{5} - 4\right) + i\left(\frac{2}{5} - \frac{5}{2}\right) \\ &= \frac{-19}{5} + i\left(\frac{-21}{10}\right) \\ &= \frac{-19}{5} - \frac{21}{10}i \end{aligned}$$

Question 7:

Express the given complex number in the form $a + ib$:

$$\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$$

Answer 7:

$$\begin{aligned} & \left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right) \\ &= \frac{1}{3} + \frac{7}{3}i + 4 + \frac{1}{3}i + \frac{4}{3} - i \\ &= \left(\frac{1}{3} + 4 + \frac{4}{3}\right) + i\left(\frac{7}{3} + \frac{1}{3} - 1\right) \\ &= \frac{17}{3} + i\frac{5}{3} \end{aligned}$$

Question 8:

Express the given complex number in the form $a + ib$: $(1 - i)^4$

Answer 8:

$$\begin{aligned}
 (1-i)^4 &= \left[(1-i)^2 \right]^2 \\
 &= \left[1^2 + i^2 - 2i \right]^2 \\
 &= \left[1 - 1 - 2i \right]^2 \\
 &= (-2i)^2 \\
 &= (-2i) \times (-2i) \\
 &= 4i^2 = -4 \qquad [i^2 = -1]
 \end{aligned}$$

Question 9:

Express the given complex number in the form $a + ib$: $\left(\frac{1}{3} + 3i\right)^3$

Answer 9:

$$\begin{aligned}
 \left(\frac{1}{3} + 3i\right)^3 &= \left(\frac{1}{3}\right)^3 + (3i)^3 + 3\left(\frac{1}{3}\right)(3i)\left(\frac{1}{3} + 3i\right) \\
 &= \frac{1}{27} + 27i^3 + 3i\left(\frac{1}{3} + 3i\right) \\
 &= \frac{1}{27} + 27(-i) + i + 9i^2 \qquad [i^3 = -i] \\
 &= \frac{1}{27} - 27i + i - 9 \qquad [i^2 = -1] \\
 &= \left(\frac{1}{27} - 9\right) + i(-27 + 1) \\
 &= \frac{-242}{27} - 26i
 \end{aligned}$$

Question 10:

Express the given complex number in the form $a + ib$: $\left(-2 - \frac{1}{3}i\right)^3$

Answer 10:

$$\begin{aligned}
 \left(-2 - \frac{1}{3}i\right)^3 &= (-1)^3 \left(2 + \frac{1}{3}i\right)^3 \\
 &= - \left[2^3 + \left(\frac{i}{3}\right)^3 + 3(2)\left(\frac{i}{3}\right)\left(2 + \frac{i}{3}\right) \right] \\
 &= - \left[8 + \frac{i^3}{27} + 2i \left(2 + \frac{i}{3}\right) \right] \\
 &= - \left[8 - \frac{i}{27} + 4i + \frac{2i^2}{3} \right] \quad [i^3 = -i] \\
 &= - \left[8 - \frac{i}{27} + 4i - \frac{2}{3} \right] \quad [i^2 = -1] \\
 &= - \left[\frac{22}{3} + \frac{107i}{27} \right] \\
 &= -\frac{22}{3} - \frac{107}{27}i
 \end{aligned}$$

Question 11:

Find the multiplicative inverse of the complex number $4 - 3i$.

Answer 11:

Let $z = 4 - 3i$

Then,

$$\bar{z} = 4 + 3i \text{ and } |z|^2 = 4^2 + (-3)^2 = 16 + 9 = 25$$

Therefore, the multiplicative inverse of $4 - 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4 + 3i}{25} = \frac{4}{25} + \frac{3}{25}i$$

Question 12:

Find the multiplicative inverse of the complex number $\sqrt{5}+3i$

Answer 12:

Let $z = \sqrt{5}+3i$

Then, $\bar{z} = \sqrt{5}-3i$ and $|z|^2 = (\sqrt{5})^2 + 3^2 = 5+9=14$

Therefore, the multiplicative inverse of $\sqrt{5}+3i$

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5}-3i}{14} = \frac{\sqrt{5}}{14} - \frac{3i}{14}$$

Question 13:

Find the multiplicative inverse of the complex number $-i$

Answer 13:

Let $z = -i$

Then, $\bar{z} = i$ and $|z|^2 = 1^2 = 1$

Therefore, the multiplicative inverse of $-i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{i}{1} = i$$

Question 14:

Express the following expression in the form of $a + ib$.

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$$

Answer 14:

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$$

$$= \frac{(3)^2 - (i\sqrt{5})^2}{\sqrt{3} + \sqrt{2}i - \sqrt{3} + \sqrt{2}i}$$

$$[(a+b)(a-b) = a^2 - b^2]$$

$$= \frac{9 - 5i^2}{2\sqrt{2}i}$$

$$= \frac{9 - 5(-1)}{2\sqrt{2}i}$$

$$[i^2 = -1]$$

$$= \frac{9 + 5}{2\sqrt{2}i} \times \frac{i}{i}$$

$$= \frac{14i}{2\sqrt{2}i^2}$$

$$= \frac{14i}{2\sqrt{2}(-1)}$$

$$= \frac{-7i}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{-7\sqrt{2}i}{2}$$

Mathematics

(Chapter – 5) (Complex Numbers and Quadratic Equations)

(Class – XI)

Exercise 5.2

Question 1:

Find the modulus and the argument of the complex number $z = -1 - i\sqrt{3}$

Answer 1:

$$z = -1 - i\sqrt{3}$$

$$\text{Let } r \cos \theta = -1 \text{ and } r \sin \theta = -\sqrt{3}$$

On squaring and adding, we obtain

$$(r \cos \theta)^2 + (r \sin \theta)^2 = (-1)^2 + (-\sqrt{3})^2$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 3$$

$$\Rightarrow r^2 = 4 \quad [\cos^2 \theta + \sin^2 \theta = 1]$$

$$\Rightarrow r = \sqrt{4} = 2 \quad [\text{Conventionally, } r > 0]$$

$$\therefore \text{Modulus} = 2$$

$$\therefore 2 \cos \theta = -1 \text{ and } 2 \sin \theta = -\sqrt{3}$$

$$\Rightarrow \cos \theta = \frac{-1}{2} \text{ and } \sin \theta = \frac{-\sqrt{3}}{2}$$

Since both the values of $\sin \theta$ and $\cos \theta$ are negative and $\sin \theta$ and $\cos \theta$ are negative in

III quadrant,

$$\text{Argument} = -\left(\pi - \frac{\pi}{3}\right) = \frac{-2\pi}{3}$$

Thus, the modulus and argument of the complex number $-1 - i\sqrt{3}$ are 2 and $\frac{-2\pi}{3}$ respectively.

Question 2:

Find the modulus and the argument of the complex number $z = -\sqrt{3} + i$

Answer 2:

$$z = -\sqrt{3} + i$$

$$\text{Let } r \cos \theta = -\sqrt{3} \text{ and } r \sin \theta = 1$$

On squaring and adding, we obtain

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (-\sqrt{3})^2 + 1^2$$

$$\Rightarrow r^2 = 3 + 1 = 4$$

$$[\cos^2 \theta + \sin^2 \theta = 1]$$

$$\Rightarrow r = \sqrt{4} = 2$$

$$[\text{Conventionally, } r > 0]$$

$$\therefore \text{Modulus} = 2$$

$$\therefore 2 \cos \theta = -\sqrt{3} \text{ and } 2 \sin \theta = 1$$

$$\Rightarrow \cos \theta = \frac{-\sqrt{3}}{2} \text{ and } \sin \theta = \frac{1}{2}$$

$$\therefore \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$[\text{As } \theta \text{ lies in the II quadrant}]$$

Thus, the modulus and argument of the complex number $-\sqrt{3} + i$ are 2 and $\frac{5\pi}{6}$ respectively.

Question 3:

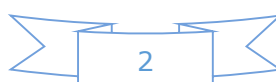
Convert the given complex number in polar form: $1 - i$

Answer 3:

$$1 - i$$

$$\text{Let } r \cos \theta = 1 \text{ and } r \sin \theta = -1$$

On squaring and adding, we obtain



$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 1^2 + (-1)^2$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$\Rightarrow r^2 = 2$$

$$\Rightarrow r = \sqrt{2} \quad [\text{Conventionally, } r > 0]$$

$$\therefore \sqrt{2} \cos \theta = 1 \text{ and } \sqrt{2} \sin \theta = -1$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{2}} \text{ and } \sin \theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \theta = -\frac{\pi}{4} \quad [\text{As } \theta \text{ lies in the IV quadrant}]$$

$$\therefore 1 - i = r \cos \theta + i r \sin \theta = \sqrt{2} \cos \left(-\frac{\pi}{4} \right) + i \sqrt{2} \sin \left(-\frac{\pi}{4} \right) = \sqrt{2} \left[\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right]$$

This is the required polar form.

Question 4:

Convert the given complex number in polar form: $-1 + i$

Answer 4:

$$-1 + i$$

$$\text{Let } r \cos \theta = -1 \text{ and } r \sin \theta = 1$$

On squaring and adding, we obtain

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (-1)^2 + 1^2$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$\Rightarrow r^2 = 2$$

$$\Rightarrow r = \sqrt{2} \quad [\text{Conventionally, } r > 0]$$

$$\therefore \sqrt{2} \cos \theta = -1 \text{ and } \sqrt{2} \sin \theta = 1$$

$$\Rightarrow \cos \theta = -\frac{1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in the II quadrant}]$$

It can be written,

$$\therefore -1+i = r \cos \theta + i r \sin \theta = \sqrt{2} \cos \frac{3\pi}{4} + i \sqrt{2} \sin \frac{3\pi}{4} = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

This is the required polar form.

Question 5:

Convert the given complex number in polar form: $-1 - i$

Answer 5:

$$-1 - i$$

Let $r \cos \theta = -1$ and $r \sin \theta = -1$

On squaring and adding, we obtain

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (-1)^2 + (-1)^2$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$\Rightarrow r^2 = 2$$

$$\Rightarrow r = \sqrt{2} \quad [\text{Conventionally, } r > 0]$$

$$\therefore \sqrt{2} \cos \theta = -1 \text{ and } \sqrt{2} \sin \theta = -1$$

$$\Rightarrow \cos \theta = -\frac{1}{\sqrt{2}} \text{ and } \sin \theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \theta = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in the III quadrant}]$$

$$\therefore -1 - i = r \cos \theta + i r \sin \theta = \sqrt{2} \cos \frac{-3\pi}{4} + i \sqrt{2} \sin \frac{-3\pi}{4} = \sqrt{2} \left(\cos \frac{-3\pi}{4} + i \sin \frac{-3\pi}{4} \right)$$

This is the required polar form.

Question 6:

Convert the given complex number in polar form: -3

Answer 6:

-3

Let $r \cos \theta = -3$ and $r \sin \theta = 0$

On squaring and adding, we obtain

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (-3)^2$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 9$$

$$\Rightarrow r^2 = 9$$

$$\Rightarrow r = \sqrt{9} = 3 \quad [\text{Conventionally, } r > 0]$$

$$\therefore 3 \cos \theta = -3 \text{ and } 3 \sin \theta = 0$$

$$\Rightarrow \cos \theta = -1 \text{ and } \sin \theta = 0$$

$$\therefore \theta = \pi$$

$$\therefore -3 = r \cos \theta + i r \sin \theta = 3 \cos \pi + i 3 \sin \pi = 3 (\cos \pi + i \sin \pi)$$

This is the required polar form.

Question 7:

Convert the given complex number in polar form: $\sqrt{3} + i$

Answer 7:

$\sqrt{3} + i$

Let $r \cos \theta = \sqrt{3}$ and $r \sin \theta = 1$

On squaring and adding, we obtain

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = (\sqrt{3})^2 + 1^2$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 3 + 1$$

$$\Rightarrow r^2 = 4$$

$$\Rightarrow r = \sqrt{4} = 2 \quad [\text{Conventionally, } r > 0]$$

$$\therefore 2 \cos \theta = \sqrt{3} \text{ and } 2 \sin \theta = 1$$

$$\Rightarrow \cos \theta = \frac{\sqrt{3}}{2} \text{ and } \sin \theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6} \quad [\text{As } \theta \text{ lies in the I quadrant}]$$

$$\therefore \sqrt{3} + i = r \cos \theta + i r \sin \theta = 2 \cos \frac{\pi}{6} + i 2 \sin \frac{\pi}{6} = 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

This is the required polar form.

Question 8:

Convert the given complex number in polar form: i

Answer 8:

i

Let $r \cos \theta = 0$ and $r \sin \theta = 1$

On squaring and adding, we obtain

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 0^2 + 1^2$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 1$$

$$\Rightarrow r^2 = 1$$

$$\Rightarrow r = \sqrt{1} = 1 \quad [\text{Conventionally, } r > 0]$$

$$\therefore \cos \theta = 0 \text{ and } \sin \theta = 1$$

$$\therefore \theta = \frac{\pi}{2}$$

$$\therefore i = r \cos \theta + i r \sin \theta = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

This is the required polar form.

Mathematics

(Chapter – 5) (Complex Numbers and Quadratic Equations)

(Class – XI)

Exercise 5.3

Question 1:

Solve the equation $x^2 + 3 = 0$

Answer 1:

The given quadratic equation is $x^2 + 3 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,
we obtain $a = 1$, $b = 0$, and $c = 3$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = 0^2 - 4 \times 1 \times 3 = -12$$

Therefore, the required solutions are

$$\begin{aligned} \frac{-b \pm \sqrt{D}}{2a} &= \frac{\pm \sqrt{-12}}{2 \times 1} = \frac{\pm \sqrt{12}i}{2} \quad [\sqrt{-1} = i] \\ &= \frac{\pm 2\sqrt{3}i}{2} = \pm \sqrt{3}i \end{aligned}$$

Question 2:

Solve the equation $2x^2 + x + 1 = 0$

Answer 2:

The given quadratic equation is $2x^2 + x + 1 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,
we obtain $a = 2$, $b = 1$, and $c = 1$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = 1^2 - 4 \times 2 \times 1 = 1 - 8 = -7$$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times 2} = \frac{-1 \pm \sqrt{7}i}{4} \quad [\sqrt{-1} = i]$$

Question 3:

Solve the equation $x^2 + 3x + 9 = 0$

Answer 3:

The given quadratic equation is $x^2 + 3x + 9 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,
we obtain $a = 1$, $b = 3$, and $c = 9$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 9 = 9 - 36 = -27$$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-27}}{2(1)} = \frac{-3 \pm 3\sqrt{-3}}{2} = \frac{-3 \pm 3\sqrt{3}i}{2} \quad [\sqrt{-1} = i]$$

Question 4:

Solve the equation $-x^2 + x - 2 = 0$

Answer 4:

The given quadratic equation is $-x^2 + x - 2 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,
we obtain $a = -1$, $b = 1$, and $c = -2$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = 1^2 - 4 \times (-1) \times (-2) = 1 - 8 = -7$$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times (-1)} = \frac{-1 \pm \sqrt{7}i}{-2} \quad [\sqrt{-1} = i]$$

Question 5:

Solve the equation $x^2 + 3x + 5 = 0$

Answer 5:

The given quadratic equation is $x^2 + 3x + 5 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain $a = 1$, $b = 3$, and $c = 5$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 5 = 9 - 20 = -11$$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-11}}{2 \times 1} = \frac{-3 \pm \sqrt{11}i}{2} \quad [\sqrt{-1} = i]$$

Question 6:

Solve the equation $x^2 - x + 2 = 0$

Answer 6:

The given quadratic equation is $x^2 - x + 2 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain $a = 1$, $b = -1$, and $c = 2$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-1)^2 - 4 \times 1 \times 2 = 1 - 8 = -7$$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-1) \pm \sqrt{-7}}{2 \times 1} = \frac{1 \pm \sqrt{7}i}{2} \quad [\sqrt{-1} = i]$$

Question 7:

Solve the equation $\sqrt{2}x^2 + x + \sqrt{2} = 0$

Answer 7:

The given quadratic equation is $\sqrt{2}x^2 + x + \sqrt{2} = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$$a = \sqrt{2}, b = 1, \text{ and } c = \sqrt{2}$$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = 1^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times \sqrt{2}} = \frac{-1 \pm \sqrt{7}i}{2\sqrt{2}} \quad \left[\sqrt{-1} = i \right]$$

Question 8:

Solve the equation $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Answer 8:

The given quadratic equation is $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$$a = \sqrt{3}, b = -\sqrt{2} \text{ and } c = 3\sqrt{3}$$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-\sqrt{2})^2 - 4(\sqrt{3})(3\sqrt{3}) = 2 - 36 = -34$$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-\sqrt{2}) \pm \sqrt{-34}}{2 \times \sqrt{3}} = \frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}} \quad \left[\sqrt{-1} = i \right]$$

Question 9:

Solve the equation $x^2 + x + \frac{1}{\sqrt{2}} = 0$

Answer 9:

The given quadratic equation is $x^2 + x + \frac{1}{\sqrt{2}} = 0$

This equation can also be written as $\sqrt{2}x^2 + \sqrt{2}x + 1 = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$$a = \sqrt{2}, b = \sqrt{2} \text{ and } c = 1$$

$$\therefore \text{Discriminant (D)} = b^2 - 4ac = (\sqrt{2})^2 - 4 \times (\sqrt{2}) \times 1 = 2 - 4\sqrt{2}$$

Therefore, the required solutions are

$$\begin{aligned} \frac{-b \pm \sqrt{D}}{2a} &= \frac{-\sqrt{2} \pm \sqrt{2-4\sqrt{2}}}{2 \times \sqrt{2}} = \frac{-\sqrt{2} \pm \sqrt{2(1-2\sqrt{2})}}{2\sqrt{2}} \\ &= \left(\frac{-\sqrt{2} \pm \sqrt{2}(\sqrt{2\sqrt{2}-1})i}{2\sqrt{2}} \right) \quad [\sqrt{-1} = i] \\ &= \frac{-1 \pm (\sqrt{2\sqrt{2}-1})i}{2} \end{aligned}$$

Question 10:

Solve the equation $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

Answer 10:

The given quadratic equation is $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

This equation can also be written as $\sqrt{2}x^2 + x + \sqrt{2} = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$$a = \sqrt{2}, b = 1 \text{ and } c = \sqrt{2}$$

$$\therefore \text{Discriminant (D)} = b^2 - 4ac = 1^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}} = \frac{-1 \pm \sqrt{7}i}{2\sqrt{2}} \quad [\sqrt{-1} = i]$$

Mathematics

(Chapter – 5) (Complex Numbers and Quadratic Equations)

(Class – XI)

Miscellaneous Exercise on chapter 5

Question 1:

Evaluate: $\left[i^{18} + \left(\frac{1}{i} \right)^{25} \right]^3$

Answer 1:

$$\begin{aligned} & \left[i^{18} + \left(\frac{1}{i} \right)^{25} \right]^3 \\ &= \left[i^{4 \times 4 + 2} + \frac{1}{i^{4 \times 6 + 1}} \right]^3 \\ &= \left[(i^4)^4 \cdot i^2 + \frac{1}{(i^4)^6 \cdot i} \right]^3 \\ &= \left[i^2 + \frac{1}{i} \right]^3 & [i^4 = 1] \\ &= \left[-1 + \frac{1}{i} \times \frac{i}{i} \right]^3 & [i^2 = -1] \\ &= \left[-1 + \frac{i}{i^2} \right]^3 \\ &= [-1 - i]^3 \\ &= (-1)^3 [1 + i]^3 \\ &= -[1^3 + i^3 + 3 \cdot 1 \cdot i(1 + i)] \\ &= -[1 + i^3 + 3i + 3i^2] \\ &= -[1 - i + 3i - 3] \\ &= -[-2 + 2i] \\ &= 2 - 2i \end{aligned}$$

Question 2:

For any two complex numbers z_1 and z_2 , prove that

$$\operatorname{Re}(z_1 z_2) = \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2$$

Answer 2:

$$\text{Let } z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

$$\therefore z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$$

$$= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2$$

$$= x_1 x_2 + ix_1 y_2 + iy_1 x_2 - y_1 y_2 \quad [i^2 = -1]$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

$$\Rightarrow \operatorname{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2$$

$$\Rightarrow \operatorname{Re}(z_1 z_2) = \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2$$

Hence, proved.

Question 3:

Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.

Answer 3:

$$\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \left[\frac{(1+i) - 2(1-4i)}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i}\right]$$

$$= \left[\frac{1+i-2+8i}{1+i-4i-4i^2}\right]\left[\frac{3-4i}{5+i}\right] = \left[\frac{-1+9i}{5-3i}\right]\left[\frac{3-4i}{5+i}\right]$$

$$= \left[\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}\right] = \frac{33+31i}{28-10i} = \frac{33+31i}{2(14-5i)}$$

$$= \frac{(33+31i)}{2(14-5i)} \times \frac{(14+5i)}{(14+5i)} \quad \left[\text{On multiplying numerator and denominator by } (14 + 5i)\right]$$

$$= \frac{462+165i+434i+155i^2}{2[(14)^2 - (5i)^2]} = \frac{307+599i}{2(196-25i^2)}$$

$$= \frac{307+599i}{2(221)} = \frac{307+599i}{442} = \frac{307}{442} + \frac{599i}{442}$$

This is the required standard form.

Question 4:

If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ prove that $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$

Answer 4:

$$\begin{aligned}x - iy &= \sqrt{\frac{a-ib}{c-id}} \\&= \sqrt{\frac{a-ib}{c-id} \times \frac{c+id}{c+id}} \quad [\text{On multiplying numerator and denominator by } (c+id)] \\&= \sqrt{\frac{(ac+bd) + i(ad-bc)}{c^2+d^2}}\end{aligned}$$

$$\begin{aligned}\therefore (x - iy)^2 &= \frac{(ac+bd) + i(ad-bc)}{c^2+d^2} \\ \Rightarrow x^2 - y^2 - 2ixy &= \frac{(ac+bd) + i(ad-bc)}{c^2+d^2}\end{aligned}$$

On comparing real and imaginary parts, we obtain

$$x^2 - y^2 = \frac{ac+bd}{c^2+d^2}, \quad -2xy = \frac{ad-bc}{c^2+d^2} \quad (1)$$

$$\begin{aligned}(x^2 + y^2)^2 &= (x^2 - y^2)^2 + 4x^2y^2 \\&= \left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{ad-bc}{c^2+d^2}\right)^2 \quad [\text{Using (1)}] \\&= \frac{a^2c^2 + b^2d^2 + 2acbd + a^2d^2 + b^2c^2 - 2adbc}{(c^2+d^2)^2} \\&= \frac{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}{(c^2+d^2)^2} \\&= \frac{a^2(c^2+d^2) + b^2(c^2+d^2)}{(c^2+d^2)^2} \\&= \frac{(c^2+d^2)(a^2+b^2)}{(c^2+d^2)^2} \\&= \frac{a^2+b^2}{c^2+d^2}\end{aligned}$$

Hence, proved.

Question 5:

Convert the following in the polar form:

(i) $\frac{1+7i}{(2-i)^2}$, (ii) $\frac{1+3i}{1-2i}$

Answer 5:

(i) Here, $z = \frac{1+7i}{(2-i)^2}$

$$= \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i} = \frac{1+7i}{4-1-4i}$$

$$= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{3^2+4^2}$$

$$= \frac{3+4i+21i-28}{3^2+4^2} = \frac{-25+25i}{25}$$

$$= -1+i$$

Let $r \cos \theta = -1$ and $r \sin \theta = 1$

On squaring and adding, we

obtain $r^2 (\cos^2 \theta + \sin^2 \theta) = 1$

+ 1

$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 2$

$\Rightarrow r^2 = 2$ $[\cos^2 \theta + \sin^2 \theta = 1]$

$\Rightarrow r = \sqrt{2}$ $[\text{Conventionally, } r > 0]$

$\therefore \sqrt{2} \cos \theta = -1$ and $\sqrt{2} \sin \theta = 1$

$\Rightarrow \cos \theta = \frac{-1}{\sqrt{2}}$ and $\sin \theta = \frac{1}{\sqrt{2}}$

$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ $[\text{As } \theta \text{ lies in II quadrant}]$

$\therefore z = r \cos \theta + i r \sin \theta$

$$= \sqrt{2} \cos \frac{3\pi}{4} + i\sqrt{2} \sin \frac{3\pi}{4} = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

This is the required polar form.

(ii) Here, $z = \frac{1+3i}{1-2i}$

$$= \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{1+2i+3i-6}{1+4}$$

$$= \frac{-5+5i}{5} = -1+i$$

Let $r \cos \theta = -1$ and $r \sin \theta$

$= 1$ On squaring and adding,

we obtain $r^2 (\cos^2 \theta + \sin^2 \theta)$

$$= 1 + 1$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 2$$

$$\Rightarrow r^2 = 2 \quad [\cos^2 \theta + \sin^2 \theta = 1]$$

$$\Rightarrow r = \sqrt{2} \quad [\text{Conventionally, } r > 0]$$

$$\therefore \sqrt{2} \cos \theta = -1 \text{ and } \sqrt{2} \sin \theta = 1$$

$$\Rightarrow \cos \theta = \frac{-1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in II quadrant}]$$

$$\therefore z = r \cos \theta + i r \sin \theta$$

$$= \sqrt{2} \cos \frac{3\pi}{4} + i\sqrt{2} \sin \frac{3\pi}{4} = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

This is the required polar form.

Question 6:

Solve the equation $3x^2 - 4x + \frac{20}{3} = 0$

Answer 6:

The given quadratic equation is $3x^2 - 4x + \frac{20}{3} = 0$

This equation can also be written as $9x^2 - 12x + 20 = 0$

On comparing this equation with $ax^2 + bx + c = 0$,
we obtain $a = 9$, $b = -12$, and $c = 20$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20 = 144 - 720 = -576$$

Therefore, the required solutions are

$$\begin{aligned} \frac{-b \pm \sqrt{D}}{2a} &= \frac{-(-12) \pm \sqrt{-576}}{2 \times 9} = \frac{12 \pm \sqrt{576}i}{18} \quad [\sqrt{-1} = i] \\ &= \frac{12 \pm 24i}{18} = \frac{6(2 \pm 4i)}{18} = \frac{2 \pm 4i}{3} = \frac{2}{3} \pm \frac{4}{3}i \end{aligned}$$

Question 7:

Solve the equation $x^2 - 2x + \frac{3}{2} = 0$

Answer 7:

The given quadratic equation is $x^2 - 2x + \frac{3}{2} = 0$

This equation can also be written as $2x^2 - 4x + 3 = 0$

On comparing this equation with $ax^2 + bx + c = 0$,
we obtain $a = 2$, $b = -4$, and $c = 3$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-4)^2 - 4 \times 2 \times 3 = 16 - 24 = -8$$

Therefore, the required solutions are

$$\begin{aligned}\frac{-b \pm \sqrt{D}}{2a} &= \frac{-(-4) \pm \sqrt{-8}}{2 \times 2} = \frac{4 \pm 2\sqrt{2}i}{4} \quad [\sqrt{-1} = i] \\ &= \frac{2 \pm \sqrt{2}i}{2} = 1 \pm \frac{\sqrt{2}}{2}i\end{aligned}$$

Question 8:

Solve the equation $27x^2 - 10x + 1 = 0$

Answer 8:

The given quadratic equation is $27x^2 - 10x + 1 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain $a = 27$, $b = -10$, and $c = 1$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1 = 100 - 108 = -8$$

Therefore, the required solutions are

$$\begin{aligned}\frac{-b \pm \sqrt{D}}{2a} &= \frac{-(-10) \pm \sqrt{-8}}{2 \times 27} = \frac{10 \pm 2\sqrt{2}i}{54} \quad [\sqrt{-1} = i] \\ &= \frac{5 \pm \sqrt{2}i}{27} = \frac{5}{27} \pm \frac{\sqrt{2}}{27}i\end{aligned}$$

Question 9:

Solve the equation $21x^2 - 28x + 10 = 0$

Answer 9:

The given quadratic equation is $21x^2 - 28x + 10 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain $a = 21$, $b = -28$, and $c = 10$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-28)^2 - 4 \times 21 \times 10 = 784 - 840 = -56$$

Therefore, the required solutions are

$$\begin{aligned}\frac{-b \pm \sqrt{D}}{2a} &= \frac{-(-28) \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm \sqrt{56}i}{42} \\ &= \frac{28 \pm 2\sqrt{14}i}{42} = \frac{28}{42} \pm \frac{2\sqrt{14}}{42}i = \frac{2}{3} \pm \frac{\sqrt{14}}{21}i\end{aligned}$$

Question 10:

If . $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right|$

Answer 10:

$$z_1 = 2 - i, z_2 = 1 + i$$

$$\begin{aligned}\therefore \left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right| &= \left| \frac{(2 - i) + (1 + i) + 1}{(2 - i) - (1 + i) + i} \right| \\ &= \left| \frac{4}{2 - 2i} \right| = \left| \frac{4}{2(1 - i)} \right| \\ &= \left| \frac{2}{1 - i} \times \frac{1 + i}{1 + i} \right| = \left| \frac{2(1 + i)}{1^2 - i^2} \right| \\ &= \left| \frac{2(1 + i)}{1 + 1} \right| \quad [i^2 = -1] \\ &= \left| \frac{2(1 + i)}{2} \right| \\ &= |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}\end{aligned}$$

Thus, the value of $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right|$ is $\sqrt{2}$.

Question 11:

If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$

Answer 11:

$$\begin{aligned} a + ib &= \frac{(x+i)^2}{2x^2+1} \\ &= \frac{x^2 + i^2 + 2xi}{2x^2+1} \\ &= \frac{x^2 - 1 + i2x}{2x^2+1} \\ &= \frac{x^2-1}{2x^2+1} + i\left(\frac{2x}{2x^2+1}\right) \end{aligned}$$

On comparing real and imaginary parts, we obtain

$$\begin{aligned} a &= \frac{x^2-1}{2x^2+1} \text{ and } b = \frac{2x}{2x^2+1} \\ \therefore a^2 + b^2 &= \left(\frac{x^2-1}{2x^2+1}\right)^2 + \left(\frac{2x}{2x^2+1}\right)^2 \\ &= \frac{x^4 + 1 - 2x^2 + 4x^2}{(2x^2+1)^2} \\ &= \frac{x^4 + 1 + 2x^2}{(2x^2+1)^2} \\ &= \frac{(x^2+1)^2}{(2x^2+1)^2} \\ \therefore a^2 + b^2 &= \frac{(x^2+1)^2}{(2x^2+1)^2} \end{aligned}$$

Hence, proved.

Question 12:

Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

(i) $\operatorname{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$, (ii) $\operatorname{Im}\left(\frac{1}{z_1 \bar{z}_1}\right)$

Answer 12:

$$z_1 = 2 - i, z_2 = -2 + i$$

$$(i) z_1 z_2 = (2 - i)(-2 + i) = -4 + 2i + 2i - i^2 = -4 + 4i - (-1) = -3 + 4i$$

$$\bar{z}_1 = 2 + i$$

$$\therefore \frac{z_1 z_2}{\bar{z}_1} = \frac{-3 + 4i}{2 + i}$$

On multiplying numerator and denominator by $(2 - i)$, we obtain

$$\begin{aligned} \frac{z_1 z_2}{\bar{z}_1} &= \frac{(-3 + 4i)(2 - i)}{(2 + i)(2 - i)} = \frac{-6 + 3i + 8i - 4i^2}{2^2 + 1^2} = \frac{-6 + 11i - 4(-1)}{2^2 + 1^2} \\ &= \frac{-2 + 11i}{5} = \frac{-2}{5} + \frac{11}{5}i \end{aligned}$$

On comparing real parts, we obtain

$$\operatorname{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \frac{-2}{5}$$

$$(ii) \frac{1}{z_1 \bar{z}_1} = \frac{1}{(2 - i)(2 + i)} = \frac{1}{(2)^2 + (1)^2} = \frac{1}{5}$$

On comparing imaginary parts, we obtain

$$\operatorname{Im}\left(\frac{1}{z_1 \bar{z}_1}\right) = 0$$

Question 13:

Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$

Answer 13:

Let $z = \frac{1+2i}{1-3i}$, then

$$\begin{aligned} z &= \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1^2+3^2} = \frac{1+5i+6(-1)}{1+9} \\ &= \frac{-5+5i}{10} = \frac{-5}{10} + \frac{5i}{10} = \frac{-1}{2} + \frac{1}{2}i \end{aligned}$$

Let $z = r \cos \theta + ir \sin \theta$

$$\text{i.e., } r \cos \theta = \frac{-1}{2} \text{ and } r \sin \theta = \frac{1}{2}$$

On squaring and adding, we obtain

$$r^2 (\cos^2 \theta + \sin^2 \theta) = \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$\Rightarrow r^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow r = \frac{1}{\sqrt{2}} \quad [\text{Conventionally, } r > 0]$$

$$\therefore \frac{1}{\sqrt{2}} \cos \theta = \frac{-1}{2} \text{ and } \frac{1}{\sqrt{2}} \sin \theta = \frac{1}{2}$$

$$\Rightarrow \cos \theta = \frac{-1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in the II quadrant}]$$

Therefore, the modulus and argument of the given complex number are

$\frac{1}{\sqrt{2}}$ and $\frac{3\pi}{4}$ respectively.

Question 14:

Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

Answer 14:

Let $z = (x - iy)(3 + 5i)$

$$z = 3x + 5xi - 3yi - 5yi^2 = 3x + 5xi - 3yi + 5y = (3x + 5y) + i(5x - 3y)$$

$$\therefore \bar{z} = (3x + 5y) - i(5x - 3y)$$

It is given that, $\bar{z} = -6 - 24i$

$$\therefore (3x + 5y) - i(5x - 3y) = -6 - 24i$$

Equating real and imaginary parts, we obtain

$$3x + 5y = -6 \quad \dots \text{ (i)}$$

$$5x - 3y = 24 \quad \dots \text{ (ii)}$$

Multiplying equation (i) by 3 and equation (ii) by 5 and then adding them, we obtain

$$9x + 15y = -18$$

$$25x - 15y = 120$$

$$\hline 34x = 102$$

$$\therefore x = \frac{102}{34} = 3$$

Putting the value of x in equation (i), we obtain

$$3(3) + 5y = -6$$

$$\Rightarrow 5y = -6 - 9 = -15$$

$$\Rightarrow y = -3$$

Thus, the values of x and y are 3 and -3 respectively.

Question 15:

Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.

Answer 15:

$$\begin{aligned}\frac{1+i}{1-i} - \frac{1-i}{1+i} &= \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} \\ &= \frac{1+i^2+2i-1-i^2+2i}{1^2+1^2} \\ &= \frac{4i}{2} = 2i \\ \therefore \left| \frac{1+i}{1-i} - \frac{1-i}{1+i} \right| &= |2i| = \sqrt{2^2} = 2\end{aligned}$$

Question 16:

If $(x + iy)^3 = u + iv$, then show that: $\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$

Answer 16:

$$\begin{aligned}(x + iy)^3 &= u + iv \\ \Rightarrow x^3 + (iy)^3 + 3 \cdot x \cdot iy(x + iy) &= u + iv \\ \Rightarrow x^3 + i^3 y^3 + 3x^2 yi + 3xy^2 i^2 &= u + iv \\ \Rightarrow x^3 - iy^3 + 3x^2 yi - 3xy^2 &= u + iv \\ \Rightarrow (x^3 - 3xy^2) + i(3x^2 y - y^3) &= u + iv\end{aligned}$$

On equating real and imaginary parts, we obtain

$$\begin{aligned}
 u &= x^3 - 3xy^2, \quad v = 3x^2y - y^3 \\
 \therefore \frac{u}{x} + \frac{v}{y} &= \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y} \\
 &= \frac{x(x^2 - 3y^2)}{x} + \frac{y(3x^2 - y^2)}{y} \\
 &= x^2 - 3y^2 + 3x^2 - y^2 \\
 &= 4x^2 - 4y^2 \\
 &= 4(x^2 - y^2) \\
 \therefore \frac{u}{x} + \frac{v}{y} &= 4(x^2 - y^2)
 \end{aligned}$$

Hence, proved.

Question 17:

If α and β are different complex numbers with $|\beta| = 1$, then find $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right|$

Answer 17:

Let $\alpha = a + ib$ and $\beta = x + iy$

It is given that, $|\beta| = 1$

$$\therefore \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow x^2 + y^2 = 1 \quad \dots (i)$$

$$\begin{aligned}
\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| &= \left| \frac{(x + iy) - (a + ib)}{1 - (a - ib)(x + iy)} \right| \\
&= \left| \frac{(x - a) + i(y - b)}{1 - (ax + aiy - ibx + by)} \right| \\
&= \left| \frac{(x - a) + i(y - b)}{(1 - ax - by) + i(bx - ay)} \right| \\
&= \frac{|(x - a) + i(y - b)|}{|(1 - ax - by) + i(bx - ay)|} \quad \left[\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right] \\
&= \frac{\sqrt{(x - a)^2 + (y - b)^2}}{\sqrt{(1 - ax - by)^2 + (bx - ay)^2}} \\
&= \frac{\sqrt{x^2 + a^2 - 2ax + y^2 + b^2 - 2by}}{\sqrt{1 + a^2x^2 + b^2y^2 - 2ax + 2abxy - 2by + b^2x^2 + a^2y^2 - 2abxy}} \\
&= \frac{\sqrt{(x^2 + y^2) + a^2 + b^2 - 2ax - 2by}}{\sqrt{1 + a^2(x^2 + y^2) + b^2(y^2 + x^2) - 2ax - 2by}} \\
&= \frac{\sqrt{1 + a^2 + b^2 - 2ax - 2by}}{\sqrt{1 + a^2 + b^2 - 2ax - 2by}} \quad [\text{Using (1)}] \\
&= 1 \\
\therefore \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| &= 1
\end{aligned}$$

Question 18:

Find the number of non-zero integral solutions of the equation $|1-i|^x = 2^x$

Answer 18:

$$\begin{aligned}
 |1-i|^x &= 2^x \\
 \Rightarrow \left(\sqrt{1^2 + (-1)^2}\right)^x &= 2^x \\
 \Rightarrow (\sqrt{2})^x &= 2^x \\
 \Rightarrow 2^{\frac{x}{2}} &= 2^x \\
 \Rightarrow \frac{x}{2} &= x \\
 \Rightarrow x &= 2x \\
 \Rightarrow 2x - x &= 0 \\
 \Rightarrow x &= 0
 \end{aligned}$$

Thus, 0 is the only integral solution of the given equation. Therefore, the number of nonzero integral solutions of the given equation is 0.

Question 19:

If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that:

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2.$$

Answer 19:

$$\begin{aligned}
 (a+ib)(c+id)(e+if)(g+ih) &= A+iB \\
 \therefore |(a+ib)(c+id)(e+if)(g+ih)| &= |A+iB| \\
 \Rightarrow |(a+ib)| \times |(c+id)| \times |(e+if)| \times |(g+ih)| &= |A+iB| \quad [|z_1 z_2| = |z_1| |z_2|] \\
 \Rightarrow \sqrt{a^2+b^2} \times \sqrt{c^2+d^2} \times \sqrt{e^2+f^2} \times \sqrt{g^2+h^2} &= \sqrt{A^2+B^2}
 \end{aligned}$$

On squaring both sides, we obtain

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2. \text{ Hence proved.}$$

Question 20:

If $\left(\frac{1+i}{1-i}\right)^m = 1$ then find the least positive integral value of m .

Answer 20:

$$\left(\frac{1+i}{1-i}\right)^m = 1$$

$$\Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$$

$$\Rightarrow \left(\frac{(1+i)^2}{1^2+1^2}\right)^m = 1$$

$$\Rightarrow \left(\frac{1^2+i^2+2i}{2}\right)^m = 1$$

$$\Rightarrow \left(\frac{1-1+2i}{2}\right)^m = 1$$

$$\Rightarrow \left(\frac{2i}{2}\right)^m = 1$$

$$\Rightarrow i^m = 1$$

$\therefore m = 4k$, where k is some integer.

Therefore, the least positive integer is 1.

Thus, the least positive integral value of m is 4 ($= 4 \times 1$).